Math 105 - Take-Home Quiz 8 - Due March 24, 2008

Instructions: You may discuss these problems with classmates, tutors, or your professor. Your write-up, however, must be your own.

Work on other paper will not be accepted. Try the problems on scrap paper, and then put your final work on this sheet. Your work should be clear, easy to follow, and without cross-outs.

1. (5 pts.) Using a telescoping sum to evaluate the following series exactly: \( \sum_{n=1}^{\infty} \frac{2}{n(n+2)} \).

\[
\frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \quad \rightarrow \quad 2 = A(n+2) + B(n) \quad \rightarrow \quad n=0: \quad 2 = 2A + 0 \rightarrow A = 1 \\
\quad n=2: \quad 2 = 0A - 2B \rightarrow B = -1.
\]

\[
S_0 \quad \sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}.
\]

\[
S_1 = \frac{1}{1} - \frac{1}{3} = \frac{2}{3}, \quad S_2 = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} = \frac{1}{2}, \quad S_3 = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} = \frac{1}{2}, \quad S_4 = \frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} = \frac{1}{2}.
\]

\[
S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}, \quad \lim_{n \to \infty} S_n = 1 + \frac{1}{2} - 0 - 0 = \frac{3}{2}.
\]

\[
S_0 \quad \sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \frac{3}{2}.
\]

2. (5 pts.) Decide whether the following series converges or diverges: \( \sum_{m=3}^{\infty} \frac{\ln m}{m^3} \). State the name of the test that you use.

Integral test. \( f(x) = \frac{\ln x}{x^3} \). \( f(x) \) is continuous and decreasing on \((3, \infty)\).

\[
\int_{3}^{\infty} \frac{\ln x}{x^3} \, dx = \lim_{t \to \infty} \left[ \frac{-\ln x}{2x^2} \right]_{3}^{t} = \lim_{t \to \infty} \left( \frac{-\ln x}{2x^2} \right)_{3}^{t} = \lim_{t \to \infty} \left( \frac{-\ln t}{2t^2} + \frac{\ln 3}{2 \cdot 3^2} - \frac{1}{3} \right) = \lim_{t \to \infty} \left( \frac{-\ln t}{2t^2} + \frac{\ln 3}{18} - \frac{1}{4t^2} + \frac{1}{4t} \right) = -\frac{1}{8} + \frac{\ln 3}{18} + \frac{1}{3}.
\]

\[(\text{Aside: } \lim_{t \to \infty} \frac{-\ln t}{2t^2} = 0, \quad \lim_{t \to \infty} \frac{\ln 3}{2t^2} = 0, \quad \lim_{t \to \infty} \frac{-1}{4t^2} = 0, \quad \lim_{t \to \infty} \frac{1}{4t} = 0) \quad \Rightarrow \quad S = 0 + \frac{\ln 3}{18} + \frac{1}{3} = \frac{\ln 3}{18} + \frac{1}{3} = \text{Finite. So the series converges!}
\]
3. (5 pts.) Decide whether the following series converges or diverges: \( \sum_{k=1}^{\infty} \frac{1}{k+2^k} \). State the name of the test that you use.

Comparison with \( \sum_{k=1}^{\infty} \frac{1}{2^k} \).

\( k+2^k > 2^k \), so \( \frac{1}{k+2^k} < \frac{1}{2^k} \).

\( \sum_{k=1}^{\infty} \frac{1}{2^k} \) is geometric with \( r = \frac{1}{2} \) (and \( a = \frac{1}{2} \)). \( 0 < r < 1 \), so it converges.

Our positive series \( \sum_{k=1}^{\infty} \frac{1}{k+2^k} \) is less than a convergent series, so our series \( \boxed{\text{converges}} \) as well.

4. (5 pts.) Decide whether the following series converges or diverges: \( \sum_{k=3}^{\infty} \frac{k!}{(3k)!} \). State the name of the test that you use.

**Ratio test:**

\[ a_k = \frac{k!}{(3k)!}, \quad a_{k+1} = \frac{(k+1)!}{(3(k+1))!} = \frac{(k+1)!}{(3k+3)!} \]

\[ \frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(3k+3)!} \cdot \frac{(3k)!}{k!} = \frac{(k+1)}{(3k+1)(3k+2)(3k+3)} \]

\[ = \frac{(k+1)}{(3k+1)(3k+2) \cdot 3(k+1)} \]

\[ \therefore L = \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{1}{(3k+1)(3k+2) \cdot 3} = 0. \]

\( \therefore L < 1 \), so the series \( \boxed{\text{converges}} \).