1.) (15 pts.) Use the Invertible Matrix Theorem to answer the following questions. In each of them, assume the matrix $A$ is $n \times n$. Also, for each of them, fully explain your reasoning.

a.) (5 pts.) If there is an $n \times n$ matrix $D$ for which $AD = I$, is there also an $n \times n$ matrix $C$ for which $CA = I$?

Yes: these are parts (k) and (j) of the IMT. They are equivalent.

b.) (5 pts.) If $A^T$ is not invertible, is $A$ invertible?

No. Parts (a) and (i.) of the IMT are "$A$ is invertible" and "$A^T$ is invertible". Either both are true, or both or false.

c.) (5 pts.) If there is a vector $b$ in $\mathbb{R}^n$ for which the equation $Ax = b$ is inconsistent, can the linear transformation $x \mapsto Ax$ be one-to-one?

The "if" part is equivalent to saying that part (g) of the IMT is false, hence all parts of the IMT is false. Then by part (f), the linear transformation $x \mapsto Ax$ cannot be 1-1.
2.) (15 pts.) Given

\[
A = \begin{bmatrix}
  1 & 2 & 0 & -1 & 3 \\
-1 & -3 & 2 & 4 & -8 \\
-2 & -1 & -6 & -7 & 9 \\
  5 & 6 & 8 & 7 & -5
\end{bmatrix},
\]

compute a basis for \( \text{Col} \ A \) and a basis for \( \text{Nul} \ A \). What is the dimension of \( \text{Col} \ A \)? What is the dimension of \( \text{Nul} \ A \)? What is \( \text{rank} \ A \)?

(Calculator): \[
A \sim \begin{bmatrix}
  1 & 0 & 4 & 5 & -7 \\
  0 & 1 & -2 & -3 & 5 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

So a basis for \( \text{Col} \ A \) is \( \begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix} \) and \( \text{dim} \ \text{Col} \ A = 2 \) (and \( \text{rank} \ A = 2 \))

Solving \( A\vec{x} = \vec{0} \), \( \vec{x} = x_3 \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ 0 \\ 0 \\ 1 \end{bmatrix} \)

So a basis for \( \text{Nul} \ A \) is \( \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix} \) and \( \text{dim} \ \text{Nul} \ A = 3 \)
3.) (15 pts.) In each of the following, assume the matrix $A$ is $3 \times 3$. Also, for each of them, fully explain your reasoning.

a.) (5 pts.) Why is $\det A^T A \geq 0$?

Since $\det (A^T) = \det A$ and $\det (AB) = (\det A)(\det B)$ (both are theorems in Chapter 3), we know

$\det A^T A = (\det A^T)(\det A) = (\det A)(\det A) = (\det A)^2$

and any number squared is $\geq 0$.

b.) (5 pts.) Is $\det(4A) = 4 \det A$?

No: $\det (4A) = 4^3 \det A$

(we can factor out a 4 from each of the 3 rows.)

c.) (5 pts.) If two rows of $A$ are the same, then what is $\det A$?

In this case, $A = \begin{bmatrix} - & - & - \\ - & - & - \\ 0 & 0 & 0 \end{bmatrix}$

and we can show this with a single row operation that does not change the determinant, so

$\det A = \det \begin{bmatrix} - & - & - \\ - & - & - \\ 0 & 0 & 0 \end{bmatrix} = 0$. 

4.) (15 pts.) Compute all eigenvalues of the matrix $A$ below. Then, for each eigenvalue, find a basis for its corresponding eigenspace.

$$ A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} $$

$$ |A - \lambda I| = \begin{vmatrix} 3 - \lambda & 2 \\ 3 & -2 - \lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) - 6 $$

$$ = \lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3) = 0 \text{ when } \lambda = -3, 4 $$

So the eigenvalues are -3 and 4.

$\lambda = -3$: $A - \lambda I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$

Solution to $(A - \lambda I) \vec{x} = \vec{0}$ are $\vec{x} = \vec{x}_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$.

Letting $\vec{x}_2 = 3$, a basis for the eigenspace is $\{\vec{e}_3\}$.

$\lambda = 4$: $A - \lambda I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

Solution to $(A - \lambda I) \vec{x} = \vec{0}$ are $\vec{x} = \vec{x}_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Letting $\vec{x}_2 = 1$, a basis for the eigenspace is $\{\vec{e}_2\}$. 
5.) (15 pts.) In each of the following, assume the matrix $A$ is $n \times n$. Also, for each of them, fully explain your reasoning.

a.) (5 pts.) If $A$ is diagonalizable, must $A$ have $n$ distinct eigenvalues?

No: it is the other way around.

(However, if $A$ has an eigenvalue of multiplicity > 1, then the corresponding eigenspace should have the same dimension, in order for $A$ to be diagonalizable.)

b.) (5 pts.) An eigenspace of $A$ is the null space of which matrix?

$$A - \lambda I$$

That is, we are finding all $\mathbf{x}$ for which

$$(A - \lambda I) \mathbf{x} = \mathbf{0}.$$ 

c.) (5 pts.) Construct a nonzero $2 \times 2$ matrix that is diagonalizable but not invertible.

(Hint: what must be true of the eigenvalues in such a matrix?)

Hint: one eigenvalue should be 0.

So, for example, 

$$\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

is diagonalizable

(in fact, it is already in diagonal form!) We can let $P = I$.

but not invertible.
6.) (15 pts.) Consider the set of vectors

\[ \{x, y, z\} = \left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \right\} \]

a.) (5 pts.) Show that \( \{x, y, z\} \) is an orthogonal set.

\[
\begin{align*}
x \cdot \bar{y} &= -\frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = 0 \\
x \cdot \bar{z} &= \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = 0 \\
y \cdot \bar{z} &= -\frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 0
\end{align*}
\]

All pairs have dot product 0, so all pairs are orthogonal, \( \therefore \) the set is orthogonal.

b.) (5 pts.) Show that \( \{x, y, z\} \) is an orthonormal set.

By part (a) all pairs are orthogonal.

Also, \( ||x|| = ||y|| = ||z|| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \sqrt{1} = 1 \). So each is a unit vector, and the set is orthonormal.

c.) (5 pts.) Compute the projection of \( y \) on the vector \( u = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} \).

\[
\text{proj}_u y = \frac{y \cdot u}{u \cdot u} u = \frac{1 + 3 + 2 - 1}{4 + 36 + 9} \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} = \frac{5}{60} \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix}
\]
7.) (10 pts.) A certain copy machine is always either working or broken (not working). If it is working today, there is a 70% chance that it will be working tomorrow. If it is broken today, there is a 50% chance that it will be broken tomorrow.

a.) (5 pts.) Assuming the copy machine is working today, what is the probability that it is working in two days?

\[
\begin{bmatrix}
.7 & .5 \\
.3 & .5
\end{bmatrix}
\]

Today

Working

Broken

\[\begin{bmatrix}
.7 & .5 \\
.3 & .5
\end{bmatrix}^2 = \begin{bmatrix}
.64 & .16 \\
.36 & .44
\end{bmatrix} \quad \text{Meaning:}

\text{The copier}

\text{is working}

\text{today.}

In two days:

\[
\begin{bmatrix}
.7 & .5 \\
.3 & .5
\end{bmatrix}^2 \begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix}
.64 & .16 \\
.36 & .44
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix}
.64 \\
.36
\end{bmatrix}
\]

\[64\%\] \quad \text{Chance it will be working in two days}

b.) (5 pts.) What is the long-term probability that the copy machine will be working on any given day? Use Markov chain techniques.

\[
\begin{bmatrix}
.7 & .5 \\
.3 & .5
\end{bmatrix}^n \rightarrow \begin{bmatrix}
1 & -0.5 \\
0 & 0.5
\end{bmatrix}
\]

Long-term: \[q_n \approx q\]. Solution for \(q\) is

\[q = x_2 \begin{bmatrix}
\frac{5}{3} \\
1
\end{bmatrix}
\]

First let \(x_2 = 3\):

\[\begin{bmatrix}
5 \\
3
\end{bmatrix}
\]

Convert to a probability vector:

\[\frac{1}{8} \begin{bmatrix}
5 \\
3
\end{bmatrix} = \begin{bmatrix}
.625 \\
.375
\end{bmatrix}
\]

Then in the long-term, there is a \(62.5\%\) chance of the copy machine working on any given day.