There are 5 total problems in this exam. On each problem, you must show all your work, or otherwise thoroughly explain your conclusions. **There is always at least one step preceding a final answer.** Units may be requested for your final answer; a point deduction will apply if they are omitted.

On the portion of the exam marked No Calculator, you will be allowed 30 minutes during which your calculator must be closed and put away. If you finish this section early, you may hand in your work early. However, **only after you hand in the “no calculators” section will you be permitted to use a calculator.** You may not return to the “no calculator” portion after handing it in.

Before beginning, ensure your calculator is set to **Radians mode.**

You are permitted up to 15 minutes before the exam to read it through without any form of writing or recording instrument. You will then have 80 minutes to complete this exam.

<table>
<thead>
<tr>
<th>Question</th>
<th>Point Value</th>
<th>Your Score</th>
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No Calculator Portion

Math 105-A (Salomone)
Exam 2
Show all your work!

Problem 1-NC. (25 points) Compute the derivative $\frac{dy}{dx}$ for each of the following relationships between $x$ and $y$. Once $\frac{dy}{dx}$ is isolated, you do not need to simplify your work any further.

(a) (8 points) $y = \sqrt{x \sin(2x)}$

$$\frac{dy}{dx} = \frac{1}{3} \left( x \sin 2x \right)^{-2/3} \left( \sin 2x + 2x \cos 2x \right)$$

(b) (8 points) $y = \frac{x + \frac{x}{1+x}}{e^x}$

$$\frac{dy}{dx} = \frac{e^x \left( 1 + \frac{1}{1+x} \right) - e^x \left( x + \ln(1+x) \right)}{(e^x)^2}$$

(c) (9 points) $\sqrt{x^2 + y^2} = 3xy$

$$\frac{1}{2} \left( x^2 + y^2 \right)^{-1/2} \left( 2x + 2y \frac{dy}{dx} \right) = 3y + 3x \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3y - \frac{x}{\sqrt{x^2 + y^2}}}{\frac{y}{\sqrt{x^2 + y^2}} - 3x}$$
Problem 2-NC. (25 points)

(a) (10 points) Compute the limit \( \lim_{x \to 0} \frac{\ln(x^2 + 1)}{e^{x^2} - 1} \), showing your work.

As \( x \to 0 \) this \( \to \frac{\ln(0^2+1)}{e^{0^2} - 1} \to 0 \). Indeterminate, so apply l'Hôpital's rule:

$$\lim_{x \to 0} \frac{\ln(x^2+1)}{e^{x^2} - 1} = \lim_{x \to 0} \frac{\frac{1}{x^2+1} \cdot 2x}{2xe^{x^2}} = \lim_{x \to 0} \frac{1}{e^{x^2}(x^2+1)} = \frac{1}{e^0(0^2+1)} = \frac{1}{1} = 1$$

(b) (15 points) Refer to the graphs of the two functions \( f(x) \) and \( g(x) \) at left. Fill in the box under the limit with a number, and the fraction boxes with either \( f(x) \) or \( g(x) \) as appropriate. Explain in one sentence why each choice is correct.

\[
\begin{align*}
\lim_{x \to 0} \frac{g(x)}{f(x)} &= 0 \\
\lim_{x \to -4} \frac{g(x)}{f(x)} &= \pm \infty \\
\lim_{x \to 0} \frac{g(x)}{f(x)} &= -2
\end{align*}
\]

Explain how you know:

If the limit is zero, then it must have had the form \( \frac{0}{\text{nonzero}} \), or possibly had that form after l'Hôpital was done.

Other options:

\( \lim_{x \to -4} \frac{g(x)}{f(x)} \) .

Explain how you know:

If the limit is \( \pm \infty \), then it must have had the form \( \frac{\text{nonzero}}{0} \) or after l'Hôpital had that form.

Other options:

\( \lim_{x \to -4} \frac{g(x)}{f(x)} \) .

Explain how you know:

Either \( g \) has twice the value of \( f \), or both \( f \) and \( g \) are zero and \( g' \) has twice the value of \( f' \).

At \( x = 0 \), \( f(0) = g(0) = 0 \) and \( f'(0) = \frac{1}{2} \), \( g'(0) = -1 \).

l'Hôpital implies

\( \lim_{x \to 0} \frac{g(x)}{f(x)} = \lim_{x \to 0} \frac{g'(x)}{f'(x)} = \frac{-1}{\frac{1}{2}} = -2 \).

Also: \( \lim_{x \to 0} \frac{g(x)}{f(x)} \).
**Problem 1.** (30 points) This problem concerns the implicit relationship graphed at left, called a *singular elliptic curve.* It satisfies the equation

\[ y^2 = x^3 - 3x + 2. \]

(a) (8 points) Solve this equation for \( y \), and by making a choice, express \( y = f(x) \) as a function of \( x \). What portion of the curve does this function’s graph make up?

\[ \text{Square root} \Rightarrow \]

\[ y = \pm \sqrt{x^3 - 3x + 2} \]

Choose \( + \) to get the function

\[ y = \sqrt{x^3 - 3x + 2} = f(x). \]

Since \( y \) is always positive or zero now, this gives the top half of the curve.

(b) (12 points) Show that this relationship satisfies the differential equation

\[ 2yy' - 3x^2 + 3 = 0. \]

It’s easiest to use the implicit equation for this purpose.

\[ y^2 = x^3 - 3x + 2 \]

\[ 2y \frac{dy}{dx} = 3x^2 - 3 \]

\[ 2y \frac{dy}{dx} - 3x^2 + 3 = 0 \]

(c) (10 points) Compute \( f'(1) \), where \( f \) is the function from part (a)?. Why does your answer make sense?

\[ f(x) = \sqrt{x^3 - 3x + 2} \]

\[ f'(x) = \frac{1}{2} (x^3 - 3x + 2)^{-1/2} (3x^2 - 3) = \frac{3}{2} \frac{x^2 - 1}{\sqrt{x^3 - 3x + 2}} \]

\[ f'(1) = \frac{3}{2} \frac{1^2 - 1}{\sqrt{1^3 - 3(1) + 2}} = \frac{3}{2} \frac{0}{0} = \text{UNDEFINED}. \]

The top half of the curve has a sharp corner at \( x = 1 \).
Problem 2. (30 points) In this problem, you will compute the derivative \( \frac{dy}{dx} \) of the function

\[ y = \arccos\left(\frac{2x}{1 + x^2}\right). \]

(a) (10 points) Rearrange the above equation in one step to eliminate the arccos. Label the triangle at left to reflect what your new equation means.

*Hint* to save you some algebra: the third side is \( 1 - x^2 \).

\[ \cos y = \frac{2x}{1 + x^2} \]

\[ \text{Adjacent} \quad \text{Hypotenuse} \]

\[ \sqrt{(1 + x^2)^2 - (2x)^2} = \sqrt{1 + 2x^2 + x^4 - 4x^2} \]

\[ = \sqrt{1 - 2x^2 + x^4} = \sqrt{(1 - x^2)^2} = |1 - x^2| \]

(b) (12 points) Compute the derivative \( \frac{dy}{dx} \) using implicit differentiation on your answer to part (a).

Your answer should still have a \( y \) in it.

\[ \frac{d}{dx} \left( \cos y = \frac{2x}{1 + x^2} \right) \]

\[ -\sin y \frac{dy}{dx} = \frac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} \]

\[ \frac{dy}{dx} = \frac{2 - 2x^2}{-\sin y (1 + x^2)^2} \]

(c) (8 points) Use your triangle from part (a) to convert all \( y \)'s in your answer to \( x \)'s.

\[ \sin y = \frac{\text{OPP}}{\text{HYP}} = \frac{1 - x^2}{1 + x^2}, \quad \text{so} \]

\[ \frac{dy}{dx} = \frac{2 - 2x^2}{1 - x^2} \cdot \frac{1}{(1 + x^2)^2} = -\frac{2(1 - x^2)(1 + x^2)}{(1 - x^2)(1 + x^2)^2} = -\frac{2}{1 + x^2}. \]
Problem 3. (40 points) A lobster fishery operating from a platform 5 miles offshore in Casco Bay wishes to run underground cables to shore to obtain high-speed internet service. The plan is to bury the wire directly from the fishery to some point along the shore, then directly along the shoreline to the nearest power substation, which is located 15 miles from the nearest point on the shore.

- Underwater cable costs $13 per mile to bury.
- Cable on shore costs only $12 per mile to bury.

How many miles of underwater cable, and how many miles of cable on shore, must be run in order for the fishery to minimize the total cost?

*Hint: try starting with the x labeled on the diagram.*

**Objective:** MINIMIZE TOTAL COST.

\[
\text{Total Cost} = \text{Underwater cost} + \text{Shore cost}
\]

Underwater cost = \( (\sqrt{x^2 + 25}) \cdot 13 = 13\sqrt{x^2 + 25} \)

Shore cost = \( (15 - x) \cdot 12 = 12(15-x) \)

Total cost \( T(x) = 13\sqrt{x^2 + 25} + 12(15-x) \)

Stationary pts: \( T'(x) = \frac{13 \cdot 2x}{2\sqrt{x^2 + 25}} - 12 = 0 \)

\[
T''(x) = \frac{13\sqrt{x^2 + 25} - \frac{13x^2}{x^2 + 25}}{x^2 + 25}
\]

\[
T''(12) = \frac{13(12^2 + 25) - 13 \cdot 12^2}{(12^2 + 25)^{3/2}} > 0 \quad \text{Loc. Min.}
\]

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<thead>
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<tbody>
<tr>
<td>0</td>
<td>$245</td>
</tr>
<tr>
<td>12</td>
<td>$205\text{.oo}</td>
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<tr>
<td>15</td>
<td>$205.55</td>
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\( 12^2 + 25 = 173 \) mi.

\( 15 - 12 = 3 \) mi.

Minimum cost is achieved with 13 mi. underwater and 3 mi. on shore.