1. Let $A = \begin{bmatrix} 3 & -8 & 14 & -1 \\ 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix}$; then $[A|I_4]$ is row-equivalent to $\begin{bmatrix} 1 & 0 & 2 & 0 & -2/7 & 4/7 & 1/7 \\ 0 & 1 & -1 & 0 & -3/7 & -1/7 & 5/7 \\ 0 & 0 & 0 & 1 & 4/7 & -1/7 & -2/7 \\ 0 & 0 & 0 & 0 & 1 & -2 & -3 & 5 \end{bmatrix}$.

Let $R = \text{rref}(A)$.

1A. Find a basis for $\text{Col}(A)$.

$rref(A)$ indicates that columns 1, 2, 4 of $A$ are pivot columns & are therefore a basis of $\text{Col}(A)$: $\{ \begin{bmatrix} 3 \\ 1 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix} \}$

1B. Find a basis for $\text{Col}(R)$.

The pivot columns of $R$ form a basis and they are (again) cols 1, 2, 4.

$\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}$

1C. Is 0 an eigenvalue of $A$? If so, find a basis for the eigenspace of 0, if 0 is not an eigenvalue, explain why not. (You do not need to find any polynomials in this problem!)

Yes, 0 is an eigenvalue of $A$ because $A\vec{x} = 0\vec{x}$, or simply, $A\vec{x} = \vec{0}$, has non-trivial solutions given by $\text{rref}(A)$ as $x_3 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ when $x_3$ is free.

$\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \}$

A basis for this eigenspace is $\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \}$

(which is also the nullspace of $A$)
This is a continuation of problem 1. For your reference we copy the following:

\[ A = \begin{bmatrix} 3 & -8 & 14 & -1 \\ 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix} \]  and \( [A | I_4] \) is row-equivalent to \( \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -2/7 & 4/7 & 1/7 \\ 0 & 1 & -1 & 0 & 0 & -3/7 & 1/7 & 5/7 \\ 0 & 0 & 0 & 1 & 0 & 4/7 & -1/7 & -2/7 \\ 0 & 0 & 0 & 0 & 1 & -2 & -3 & 5 \end{bmatrix} \).

1D. Find all conditions both necessary and sufficient for a vector \( \mathbf{b} \) with entries \( b_1, \ldots, b_4 \) to be in \( \text{Col}(A) \).

\[
\begin{align*}
\text{(the system represented by)} & \\
A\mathbf{x} = \mathbf{b} & \iff b_1 - 2b_2 - 3b_3 + 5b_4 = 0
\end{align*}
\]

1E. Find all conditions both necessary and sufficient for a vector \( \mathbf{b} \) with entries \( b_1, \ldots, b_4 \) to be in \( \text{Col}(R) \).

We need \( \text{ref}(R | I_4) \) to do this by "row-agugated matrix."

But \( \text{ref}(R | I_4) \) is \( \begin{bmatrix} R | I_4 \end{bmatrix} \) and its last row is \( \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).

So \( \mathbf{b} \in \text{Col}(R) \iff b_4 = 0 \). This is also obvious by looking at L.C.'s possible as L.C.'s of the 4 columns of \( R \), which all have 0's as their fourth entries.

1F. Find a non-zero vector which is in \( \text{Col}(A) \) but not \( \text{Col}(R) \). Show your reasoning.

Any vector \( \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \) satisfying \( \boxtimes \) but not \( b_4 = 0 \) will do. \( \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \) is an easy example.

1G. Find a non-zero vector which is in \( \text{Col}(R) \) but not \( \text{Col}(A) \). Show your reasoning.

Now we need a vector satisfying \( b_4 = 0 \) but \( \boxtimes \), and \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) is an obvious choice.

1H. Find a non-zero vector which is in both \( \text{Col}(A) \) and \( \text{Col}(R) \). Show your reasoning.

Now we require a solution of the system \( \begin{cases} b_1 - 2b_2 - 3b_3 + 5b_4 = 0 \\ b_4 = 0 \end{cases} \).

\( \begin{bmatrix} 5 \\ 0 \end{bmatrix} \) is an easy one to spot. (You could ref this system's matrix...).
2. Suppose an economy is modeled with four sectors A, B, C, and D. Suppose that sector C consumes 1/4 of every sector's output, including that of C itself. Suppose that the output of A is equally divided by and consumed by all four sectors. Suppose that D consumes half of its own output, while B uses none of its own, and none of C’s. What C and D do not consume of D’s output is divided equally and used by A and B; B and D consume equal amounts of each other’s output and this goes for C and D as well. Finally, the output of B not used by sectors B, C, and D is consumed by A and A also consumes any output of C not used by sectors B, C, and D.

2A. Find the exchange table for this economy. You may assume all columns sum to one.

\[
\begin{array}{cccc}
\frac{1}{4} & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\
\frac{1}{4} & 0 & 0 & \frac{1}{8} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{8} & \frac{1}{2} & \\
\end{array}
\]

2B. Find the complete set \( \{P_A, P_B, P_C, P_D\} \) of equilibrium prices for this economy. Write down any system of equations and augmented matrices you use in solving this problem. (Note well: if you need to enter 1/3 into your calculator as a matrix entry, do it as “1 ÷ 3” rather than entering 0.333 or some such bad decimal approximation). Use fractions in your answers, not decimals.

\[
\begin{align*}
\frac{1}{4}P_A + \frac{1}{8}P_B + \frac{1}{2}P_C + \frac{1}{8}P_D &= P_A \\
\frac{1}{4}P_A + \frac{1}{2}P_B &= P_B \\
\frac{1}{4}P_A + \frac{1}{4}P_B + \frac{1}{2}P_C + \frac{1}{2}P_D &= P_C \\
\frac{1}{4}P_A + \frac{1}{8}P_B + \frac{1}{2}P_C + \frac{1}{2}P_D &= P_D
\end{align*}
\]

\[
\begin{bmatrix}
\frac{1}{4} & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} & 0 \\
\frac{1}{4} & -1 & 0 & \frac{1}{8} & 0 \\
\frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & 0 \\
\frac{1}{4} & \frac{1}{8} & \frac{1}{4} & -\frac{1}{2} & 0 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
P_A \\
P_B \\
P_C \\
P_D
\end{bmatrix}
=
\begin{bmatrix}
75/74 \\
28/74 \\
59/74 \\
2
\end{bmatrix}
\]

where \( P_D \) is free.

2C. Suppose \( P_D \) is 100 dollars. Rank all four equilibrium prices from least to greatest.

If \( P_D = 100 \), then \( P_A \approx 101.35 \), \( P_B = 28.74 \), \( P_C = 37.84 \), \( P_D = 100 \), so

\[
P_B < P_C < P_D < P_A
\]

(and this ranking is independent of any (positive) choice of \( P_D \)).
3. Let \( A = \begin{bmatrix} -9 & 24 & 12 \\ -4 & 11 & 4 \\ 0 & 0 & 3 \end{bmatrix} \)

3A. Find the characteristic polynomial of \( A \). Show all your work. Choose your “expansion” wisely and don’t make any sign mistakes at all. Show all your work.

\[
\text{char poly}(A) = \begin{vmatrix} -9-\lambda & 24 & 12 \\ -4 & 11-\lambda & 4 \\ 0 & 0 & 3-\lambda \end{vmatrix} = \begin{vmatrix} -9-\lambda & 24 \\ -4 & 11-\lambda \end{vmatrix} (3-\lambda) = (3-\lambda)((-9-\lambda)(11-\lambda) + 96) = (3-\lambda)(\lambda^2 - 2\lambda - 99 + 96) = (3-\lambda)(\lambda^2 - 2\lambda - 3) = (3-\lambda)(\lambda - 3)(\lambda + 1) = -(\lambda - 3)^2(\lambda + 1)
\]

3B. What are the eigenvalues of \( A \), and their multiplicities?

\( \lambda = 3 \) or \( -1 \), with multiplicities 2 and 1, respectively.

3C. Find a basis for the eigenspaces of each of those eigenvalues. Organize your work nicely; make sure it’s clear which basis goes with which eigenvector.

For \( \lambda = 3 \): we need all solutions of \( A\mathbf{x} = 3\mathbf{x} \), or \( (A - 3I_3)\mathbf{x} = \mathbf{0} \), or, the nullspace of \( (A - 3I_3) \)

\[
A - 3I_3 = \begin{bmatrix} -12 & 24 & 12 \\ -4 & 8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
\Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{a basis is } \{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}
\]

For \( \lambda = -1 \), we need all solutions of \( A\mathbf{x} = -\mathbf{x} \), or \( (A + I_3)\mathbf{x} = \mathbf{0} \), or, the nullspace of \( (A + I_3) \)

\[
A + I_3 = \begin{bmatrix} -8 & 24 & 12 \\ -4 & 12 & 4 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -3 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{a basis is } \{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \}
\]

3D. Is \( A \) diagonalizable? If so find \( P \), \( D \) and \( P^{-1} \) with the appropriate properties; if \( A \) is not diagonalizable, explain why it’s not.

Yes: let \( P = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \), \( D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \) (and \( P^{-1} = \begin{bmatrix} -1 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & -2 & -1 \end{bmatrix} \)).
4. Let $B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 5 & 0 \\ -4 & -4 & -1 \end{bmatrix}$

4A. Show that $B$ has the same characteristic polynomial as does the matrix $A$ in problem 3. Show your work, make no sign errors, and write your answer in factored form:

$$\text{Char poly}(B) = (-1-\lambda) \begin{vmatrix} -1 & -2 \\ 2 & 5-\lambda \end{vmatrix} = -(1+\lambda)(1-\lambda)(5-\lambda) + 4$$
$$= -(1+\lambda)(\lambda^2 - 6\lambda + 5 + 4)$$
$$= -(1+\lambda)(\lambda^2 - 6\lambda + 9)$$
$$= -(1+\lambda)(\lambda-3)^2$$

4B. This matrix $B$ is not diagonalizable. Given this, what must be the case about one of the eigenvalues and the dimension of its eigenspace? (In your answer, say which eigenvalue has to be “at fault”).

One of the eigenvalues must have an eigenspace whose dimension is strictly less than its multiplicity, since the dimension of an eigenspace is always at least one, the only thing that can fail here is that the dimension of the eigenspace for $\lambda=3$ must be less than 2; let's see.

4C. Verify your answer to 4C by finding that eigenspace.

$$\begin{bmatrix} 1 & -3 & -2 & 0 \\ 2 & 5-3 & 0 \\ -1 & -4 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenspace for $\lambda=3$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, which has dimension 1 < multiplicity of $\lambda=3$.

4D. Even though $B$ is not diagonalizable, you should still be able to find a matrix $Q$ such that $BQ = QD$ where $D$ is the same as in problem 3. Do it.

Note: we still need an eigenvector for $\lambda=-1$, and we find $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a basis for that eigenspace. NOTE this is DIFFERENT from for $\lambda=-1$ in problem 3.

So: let $Q = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

You can CHECK that $BQ = QD$. However, since $Q^{-1}$ does NOT exist, we have NOT found a diagonalization for $B$!!!
5. Suppose that when applied in the order: “first do Op1, then do Op2”, the following row operations turn the matrix $A$ into the matrix $B = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$:

Op1: three copies of row 2 are subtracted from row 1.
Op2: row 2 is multiplied by five.

5A: What elementary matrices $F$ and $G$ represent the operations Op1 and Op2, respectively?

$$ F = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} $$

5B: Which matrix equation is true? (1) $FGA = B$ (2) $GFA = B$

5C. Find $\det(A)$.

$$ \det(G) \det(F) \det(A) = \det(B) \quad \text{(from 5B)} $$

$$ \text{so} \quad 5 \cdot 1 \cdot \det(A) = 50 $$

$$ \det(A) = 10 $$

5D. Find $A$.

Since $GFA = B$,

$$ A = F^{-1} G^{-1} B $$

$$ = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} $$

$$ = \begin{bmatrix} 5 & 6 \\ 0 & 2 \end{bmatrix} $$

5E. Find $A^{-1}$.

$$ = \frac{1}{5 \cdot 2 - 0 \cdot 6} \begin{bmatrix} 2 & 6 \\ 0 & 5 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2 & 6 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{3}{5} \\ 0 & \frac{1}{2} \end{bmatrix} $$
6. Suppose \( B = \begin{bmatrix} x & y & z \\ 3 & 4 & 6 \\ a & b & c \end{bmatrix} \) has determinant 10.

Find the determinant of each of the following matrices. You do NOT need to list any rules about matrices you used to find the det. (e.g., “swapping rows changes the sign of the det” or “the inverse of the derivative of a matrix is the matrix of its eigenvalues” this second fact is nonsense). Just FIND the determinants.

6A. \( \begin{bmatrix} x - 18 & y - 24 & z - 36 \\ 3 & 4 & 6 \\ a & b & c \end{bmatrix} \) the det is:

\[ 6A \sim B \text{ by adding a multiple of } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ to row 1} \]

\[ \det(6A) = \det(B) = 10 \]

6B. \( \begin{bmatrix} -18 & -24 & -36 \\ 3 & 4 & 6 \\ a & b & c \end{bmatrix} \) the det is:

\[ \begin{bmatrix} -18 & -24 & -36 \\ 3 & 4 & 6 \\ a & b & c \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 3 & 4 & 6 \\ a & b & c \end{bmatrix} \]

\

so \( \det(\begin{bmatrix} -18 & -24 & -36 \\ 3 & 4 & 6 \\ a & b & c \end{bmatrix}) = 0 \)

6C. \( \begin{bmatrix} x & y & z \\ 1/4 & 1/3 & 1/2 \\ a & b & c \end{bmatrix} \) the det is:

\[ \begin{bmatrix} x & y & z \\ 1/4 & 1/3 & 1/2 \\ a & b & c \end{bmatrix} \sim \begin{bmatrix} 1/4 & 1/3 & 1/2 \\ 1/4 & 1/3 & 1/2 \\ a & b & c \end{bmatrix} \]

\[ \det(B) = \det(\begin{bmatrix} 1/4 & 1/3 & 1/2 \\ 1/4 & 1/3 & 1/2 \\ a & b & c \end{bmatrix}) \]

So \( \det(B) = 12 \det(\begin{bmatrix} x & y & z \\ 1/4 & 1/3 & 1/2 \\ a & b & c \end{bmatrix}) \)

So \( 10 = 12 \det(6C) \)

So \( \det(6C) = \frac{10}{12} = \frac{5}{6} \)

6D. \( \begin{bmatrix} x & 3 & a \\ y & 4 & b \\ z & 6 & c \end{bmatrix} \) the det is:

\[ \text{This is } B^T. \]

\[ \Rightarrow \text{ its det is } \det(B) = 10 \]

6E. \( 4B \) the det is:

Since EACH row of \( B \) is multiplied by 4,

\[ \det(4B) = 4 \cdot 4 \cdot \det(B) = 640 \]

BONUS: At the review we mentioned that the intersection of two subspaces of a vector space is another subspace. Find a basis for \( \text{Col}(A) \cap \text{Col}(R) \) for \( A \) and \( R \) from problem 1. Use the back of this sheet if necessary.

\[ \text{Hint: } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \text{Col}(A) \cap \text{Col}(R) \iff b_1 \text{ satisfies both conditions for problems 10 and 1E;} \]

this gives 2 equations in 4 variables. You can solve that!