Math 205 (Winter 2011)
Test 2 (50 points)

Name: Solutions

- Check that you have 8 questions on three pages.
- Show all your work to receive full credit for a problem.

1. (5 points) Let $A$ and $B$ be $3 \times 3$ matrices, with $\det A = 4$ and $\det A^2 B^{-1} = -8$. Use properties of determinants to compute:

   (a) $\det 2A = 2^3 \det A = 8(4) = 32$.

   (b) $\det B = \det A^2 B^{-1} = -8 \cdot (\det A)(\det B^{-1}) = -8$.

   So $16 \det B^{-1} = -8 \cdot \det B^{-1} = -\frac{1}{2}$.

   $\det B = \frac{1}{\det B^{-1}} = -2$.

2. (4 points) Let $H = \text{Span} \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6 \}$ be a subspace of $\mathbb{R}^5$. Is the set $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6 \}$ a basis for $H$? Explain.

   $H$ is a subspace of $\mathbb{R}^5$.

   So $\dim H \leq 5$.

   Hence $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6 \}$ cannot be a basis for $H$ because any basis of $H$ has at most five vectors in it.

   (You can also show that the set is not linearly independent and hence not a basis.)
3. (6 points) Suppose \( \vec{u} \) is an eigenvector of a \( 4 \times 4 \) matrix \( A \) corresponding to the eigenvalue \( -3 \).

(a) Is \( A + 3I \) an invertible matrix? Explain.

Since \( -3 \) is an eigenvalue of \( A \), it satisfies the characteristic eqn. So \( \det(A+3I) = 0 \).

Hence \( A + 3I \) is not invertible.

(b) Show that \( \vec{u} \) is an eigenvector of \( A^3 \) and find the corresponding eigenvalue.

\[
A \vec{u} = -3 \vec{u}
\]

\[
A(A \vec{u}) = A(-3 \vec{u}) = -3(A\vec{u}) = -3(-3 \vec{u}) = 9 \vec{u}
\]

So \( A^2 \vec{u} = 9 \vec{u} \).

\[
A(A^2 \vec{u}) = A(9 \vec{u}) = 9(A\vec{u}) = 9(-3 \vec{u}) = -27 \vec{u}
\]

So \( A^3 \vec{u} = -27 \vec{u} \). So \( \vec{u} \) is an eigenvector of \( A^3 \) and corresponding eigenvalue is \(-27\).

4. (4 points) Determine if the following set is a subspace of the appropriate space. If the set is a subspace, find a basis and the dimension of the subspace. If the set is not a subspace, provide a counterexample to illustrate that one of the conditions in the definition of subspace does not hold.

\[
W = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \text{ where } a, b \text{ are non-negative real numbers} \right\}
\]

\( W \) is not a subspace of \( \mathbb{R}^2 \) because it is not closed under scalar multiplication.

Counterexample: \( c = -1 \), \( \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) in \( W \).

\( c \vec{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \) not in \( W \) because \(-1\) is negative.

(Any negative scalar and any vector in \( W \) would not work.)
5. (8 points) Let \( \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \) and let \( \vec{w} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \).

(a) Compute \( \left( \frac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{v}} \right) \vec{v} \) and call it \( \vec{y} \).

\[
\vec{v} \cdot \vec{w} = 3 \\
\vec{v} \cdot \vec{v} = 1+4 = 5 \\
\vec{y} = \frac{3}{5} \vec{v} = \begin{bmatrix} 3/5 \\ 0 \\ 6/5 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0 \\ 1.2 \end{bmatrix}
\]

(b) Compute \( \vec{w} - \vec{y} \) and call it \( \vec{z} \).

\[
\vec{z} = \vec{w} - \vec{y} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0.6 \\ 0 \\ 1.2 \end{bmatrix} = \begin{bmatrix} 2.4 \\ -1 \\ -1.2 \end{bmatrix}
\]

(c) Let \( L = \text{Span}\{\vec{v}\} \). Which of the two vectors \( \vec{y} \) and \( \vec{z} \) is in \( L^\perp \)? Explain.

\[
\vec{y} \cdot \vec{v} = 0.6 + 2.4 = 3 \neq 0. \quad \text{So } \vec{y} \text{ is not in } L^\perp.
\]

\[
\vec{z} \cdot \vec{v} = 2.4 - 2.4 = 0. \quad \text{So } \vec{z} \text{ is in } L^\perp.
\]

(d) Find the distance between \( \vec{v} \) and \( \vec{w} \).

\[
\text{Distance} = ||\vec{v} - \vec{w}|| = \left\| \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\| = \sqrt{4+1+4} = \sqrt{9} = 3
\]
6. (9 points) Suppose a $6 \times 6$ matrix $A$ has only three distinct eigenvalues, 1, 0 and $-1$. Suppose 
$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for the eigenspace corresponding to the eigenvalue 1 and $\{\vec{w}_1, \vec{w}_2\}$ is a 
basis for the eigenspace corresponding to the eigenvalue 0.

(a) Is $A$ diagonalizable? Explain.

dim of eigenspace corresponding to 1 is 3 and dimension
of eigenspace corresponding to 0 is 2. Dimension of
eigenspace corresponding to $-1$ is at least 1. This dimension
cannot be greater than 1 because then, combining the three
bases, we will have a linear set of more than six vectors in $\mathbb{R}^6$,
which is impossible. Thus sum of dimensions of the three eigenspaces
$= 3 + 2 + 1 = 6 = \text{number of columns of } A$. So $A$ is diagonalizable.

(b) Let $\vec{b} = 2\vec{v}_1 - 5\vec{v}_2$ be a vector in $\mathbb{R}^3$. Is $\vec{b}$ an eigenvector of $A$? Explain. If it is an
eigenvector, find the corresponding eigenvalue.

\[
A\vec{b} = A(2\vec{v}_1 - 5\vec{v}_2) = A(2\vec{v}_1) - A(5\vec{v}_2)
= 2(A\vec{v}_1) - 5(A\vec{v}_2)
= 2(A\vec{v}_1) - 5(A\vec{v}_2) \quad (A\vec{v}_1 = \vec{v}_1, A\vec{v}_2 = \vec{v}_2)
\]
So $A\vec{b} = \vec{b} \cdot \vec{b} \neq 0$, otherwise $\vec{v}_2$ would be a
Thus $\vec{b}$ is an eigenvector of $A$ and the corresponding

eigenvalue is 1.

(c) What is dim Nul $A$ and rank $A$? Explain.

Dimension of eigenspace corresponding to 0 is 2.
Eigenspace corresponding to 0 is Nul $A$ (by definition
of eigenspace).
So dim Nul $A = 2$.
By rank theorem, $6 = \text{dim Nul } A + \text{rank } A$.
So rank $A = 6 - 2 = 4$. 
7. (7 points) Let \( B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ 1 & 3 & 1 & 2 & -4 \\ 0 & -1 & 5 & -1 & 6 \\ 2 & 5 & 0 & 3 & -9 \\ 0 & 2 & 3 & 2 & 1 \end{bmatrix} \).

(a) Find a basis for \( \text{Col} \ B \) and then state the dimension of \( \text{Col} \ B \).

\[ B \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

Put pivots in first three columns. So the last two columns of \( B \) are linear combinations of the first three columns.

So basis for \( \text{Col} \ B = \{ b_1, b_2, b_3 \} \). Call it \( \beta \).

(b) Let \( \vec{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix} \). Find the coordinates of \( \vec{x} \) with respect to the basis you found in part (a).

To find coordinates of \( \vec{x} \) with respect to the basis \( \beta \), we solve the eqn. \( c_1 b_1 + c_2 b_2 + c_3 b_3 = \vec{x} \).

\[ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vec{x} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

So \( c_1 = -1, \ c_2 = 1, \ c_3 = 0 \).

Thus, \( [\vec{x}]_\beta = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \).
8. (7 points) Let $C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$.

(a) Find all the eigenvalues of $C$.

Solve $\det(C - \lambda I) = 0$. \[ \det \begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 3 & 1-\lambda \end{bmatrix} = 0 \]

First we find the determinant by expanding across second row:

\[ \det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 1-\lambda \end{bmatrix} = -0 + (1-\lambda) \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = 0 \]

\[ = (1-\lambda)(1-(\lambda)^2 - 1) = (1-\lambda)(\lambda^2 - 2\lambda) \]

So $\det(C - \lambda I) = 0$ gives $\lambda(1-\lambda)(\lambda - 2) = 0$.

Thus, $\lambda = 0, 1, 2$. Eigenvalues of $C$: 0, 1, 2.

(b) Find a basis for $\text{Nul } C$.

Solve $C\overrightarrow{x} = \overrightarrow{0}$.

$[C \ 0] \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$X_1 = -X_3$

$X_2 = 0$

$X_3$ free.

$\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Thus, $\text{Nul } C = \text{Span} \{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$ and the set is lin ind.

So basis for $\text{Nul } C = \{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$. 