Math 205 (Winter 2011)
Test 2 (50 points)

Name: ________________________________

• Check that you have 8 questions on three pages.
• Show all your work to receive full credit for a problem.

1. (5 points) Let $A$ and $B$ be $3 \times 3$ matrices, with $\det A = 4$ and $\det A^2B^{-1} = -8$. Use properties of determinants to compute:

(a) $\det 2A$

(b) $\det B$

2. (4 points) Let $H = \text{Span} \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6 \}$ be a subspace of $\mathbb{R}^6$. Is the set $\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6 \}$ a basis for $H$? Explain.
3. (6 points) Suppose $\mathbf{u}$ is an eigenvector of a $4 \times 4$ matrix $A$ corresponding to the eigenvalue $-3$.

(a) Is $A + 3I$ an invertible matrix? Explain.

(b) Show that $\mathbf{u}$ is an eigenvector of $A^3$ and find the corresponding eigenvalue.

4. (4 points) Determine if the following set is a subspace of the appropriate space. If the set is a subspace, find a basis and the dimension of the subspace. If the set is not a subspace, provide a counterexample to illustrate that one of the conditions in the definition of subspace does not hold.

$$W = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \text{ where } a, b \text{ are non-negative real numbers} \right\}.$$
5. (8 points) Let \( \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \) and let \( \vec{w} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \).

(a) Compute \( \vec{v} \cdot \vec{w} / \| \vec{v} \| \) \( \vec{v} \) and call it \( \vec{y} \).

(b) Compute \( \vec{w} - \vec{y} \) and call it \( \vec{z} \).

(c) Let \( L = \text{Span}\{\vec{v}\} \). Which of the two vectors \( \vec{y} \) and \( \vec{z} \) is in \( L^\perp \)? Explain.

(d) Find the distance between \( \vec{v} \) and \( \vec{w} \).
6. (9 points) Suppose a $6 \times 6$ matrix $A$ has only three distinct eigenvalues, 1, 0 and $-1$. Suppose
\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} is a basis for the eigenspace corresponding to the eigenvalue 1 and \{\vec{w}_1, \vec{w}_2\} is a
basis for the eigenspace corresponding to the eigenvalue 0.

(a) Is $A$ diagonalizable? Explain.

(b) Let $\vec{b} = 2\vec{v}_1 - 5\vec{v}_2$ be a vector in $\mathbb{R}^6$. Is $\vec{b}$ an eigenvector of $A$? Explain. If it is an
eigenvector, find the corresponding eigenvalue.

(c) What is $\text{dim Nul } A$ and $\text{rank } A$? Explain.
7. (7 points) Let \( B = \begin{bmatrix} 1 & 3 & 1 & 2 & -4 \\ 0 & -1 & 5 & -1 & 6 \\ 2 & 5 & 0 & 3 & -9 \\ 0 & 2 & 3 & 2 & 1 \end{bmatrix} \).

(a) Find a basis for \( \text{Col} \ B \) and then state the dimension of \( \text{Col} \ B \).

(b) Let \( \vec{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix} \). Find the coordinates of \( \vec{x} \) with respect to the basis you found in part (a).
8. (7 points) Let \( C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \).

(a) Find all the eigenvalues of \( C \).

(b) Find a basis for \( \text{Nul} \ C \).