Math 105 Test 2 (75 points)

Name: Solutions

- Check that you have 9 questions on four pages.
- Show all your work to receive full credit for a problem.

You may use any of the following facts:

\[(uv)' = u'v + uv'\]
\[\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}\]
\[(f \circ g)'(x) = f'(g(x))g'(x)\]

\[(b^x)' = (\ln b)b^x\]
\[(\log_b x)' = \frac{1}{(\ln b)x}\]

\[(\sin x)' = \cos x\]
\[(\cos x)' = -\sin x\]
\[(\tan x)' = \sec^2 x\]

\[(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}\]
\[(\arccos x)' = \frac{-1}{\sqrt{1 - x^2}}\]
\[(\arctan x)' = \frac{1}{1 + x^2}\]

1. (6 points) The Intermediate Value Theorem (IVT) says that if a function \(f\) is continuous on the interval \([a, b]\) and \(y\) is any number between \(f(a)\) and \(f(b)\), then \(f(c) = y\) for some input \(c\) in \([a, b]\).

Let \(f(x) = 2 - x + 3x^4\). Does the IVT guarantee that \(f(x)\) has a root in the interval \([-1, 1]\)? Explain.

\[f(-1) = 2 - 1 + 3 = 4\]
\[f(1) = 2 - 1 + 3 = 4.\]

0 is not between 4 and 6.

So the IVT does not guarantee that there is a \(c\) in the interval \([-1, 1]\) such that \(f(c) = 0\).

So the IVT does not guarantee that \(f\) has a root in the interval \([-1, 1]\).
2. (10 points) Find the derivative of the following functions.

(a) \( g(x) = \frac{3 \ln x}{4 - \tan(x^2)} \)

\[
g'(x) = \frac{(4 - \tan(x^2)) \left( \frac{3}{x} \right) - 3 \ln x (-\sec^2(x^2)) \cdot 2x}{(4 - \tan(x^2))^2}
\]

\[
eq \frac{3(4 - \tan(x^2))}{x} + 6 \ln x \sec^2(x^2)
\]

(b) \( f(x) = \sqrt{\arcsin(2x)} = (\arcsin(2x))^{1/2} \)

\[
f'(x) = \frac{1}{2} \left( \arcsin(2x) \right)^{-1/2} \cdot \frac{1}{\sqrt{1-(2x)^2}} \cdot 2
\]

\[
f''(x) = \frac{1}{\sqrt{\arcsin(2x)} \cdot (1-4x^2)}
\]
3. (8 points) Evaluate the following limits.

(a) \( \lim_{t \to \infty} \frac{10}{e^{2t}} = 0 \) (as seen from the graph)

Graph of \( \frac{10}{e^{2t}} \)

(b) \( \lim_{x \to 3} \frac{\sin(\pi x)}{9 - 3x} = \frac{0}{0} \)

\[
= \lim_{x \to 3} \frac{\pi \cos(\pi x)}{-3} \\
= \frac{\pi \cos(3\pi)}{-3} \\
= \frac{-\pi}{-3} = \frac{\pi}{3}.
\]
4. (8 points) Consider the curve defined by the equation \( x^2 - 5y = x^3y^2 \). Use implicit differentiation to find \( \frac{dy}{dx} \):

\[
x^2 - 5y = x^3y^2
\]

Differentiate both sides of the equation:

\[
2x - 5y' = 3x^2y^2 + x^3 \cdot 2yy'
\]

\[
2x - 3x^2y^2 = 2x^3yy' + 5y'
\]

\[
y'(2x^3y + 5) = 2x - 3x^2y^2
\]

\[
y' = \frac{2x - 3x^2y^2}{2x^3y + 5}
\]

5. (5 points) Simplify \( \cos(\arctan x) \) as much as possible.

Let \( \arctan x = \theta \). So \( \tan \theta = x \).

With this information, we can construct the following right triangle:

\[
\sqrt{1 + x^2} = c
\]

\[
\cos (\arctan x) = \cos \theta = \frac{1}{c} = \frac{1}{\sqrt{1 + x^2}}
\]
6. (10 points) Suppose the domain of \( f(x) \) is all real numbers and that \( f \) has an inverse function \( f^{-1}(x) \). A part of the graph of \( f \) on the interval \([-10, 10]\) is given below. Let \( h(x) = \frac{6}{f(x)} \).

(a) Can the point \((100, -2)\) be on the graph of \( f \)? Justify.

Since \( f \) has an inverse function, \( f \) is a one-to-one function. From the graph, we see that \( f(-3) = -2 \).

Since \( f \) is one-to-one, we cannot have \( f(100) = -2 \) as well. So the point \((100, -2)\) cannot be on the graph of \( f \).

(b) Find \( f^{-1}(-2) \).

\[ f(-3) = -2 \quad \text{So} \quad f^{-1}(-2) = -3 \]

(c) Find \( h(-2) \).

\[ h(-2) = \frac{6}{f(-2)} = \frac{6}{-3} = -2 \]

(d) Is \( h \) increasing, decreasing, or stationary at \( x = -2 \)? Justify.

\[ h(x) = \frac{6}{f(x)} = 6 (f(x))^{-1} \]

So \( h'(x) = -6 (f(x))^{-2} f'(x) \) i.e. \( h'(x) = \frac{-6 f'(x)}{(f(x))^2} \)

\[ h'(-2) = \frac{-6 f'(-2)}{(f(-2))^2} = \frac{-6 f'(-2)}{(-3)^2} = \frac{-6 f'(-2)}{9} \]

At \(-2\), \( f \) has a negative slope, i.e. \( f'(-2) < 0 \).

So \( h'(-2) = -\frac{6}{9} \cdot \) (negative number) = positive number.

Hence \( h \) is increasing at \( x = -2 \).
7. (10 points) Two identical adjacent rectangular plots of farmland are to be fenced in using 100 meters of wire fencing, as shown in the figure. What are the dimensions of the enclosure that has maximum area? What is the maximum area that can be fenced in?

\\[
\begin{align*}
\text{++++ indicates fencing.} \\
\text{We want to maximize area, } A. \\
A &= a \cdot w \cdot l \cdot \\
\text{Constraint eqn:} \\
2l + 3w &= 100 \\
2l &= 100 - 3w \\
l &= \frac{100 - 3w}{2} \\
\text{So } A &= w \left(50 - \frac{3w}{2}\right) \text{ i.e. } A(w) = 50w - \frac{3w^2}{2} \\
\text{Find maximum: } A'(w) &= 50 - 3w \\
A'(w) &= 0 \text{ gives } w = \frac{50}{3} \text{ (critical point)}.
\end{align*}
\]

<table>
<thead>
<tr>
<th>Sign of $A'$</th>
<th>+</th>
<th>-</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direction of $A$</td>
<td>$\uparrow$</td>
<td>$\downarrow$</td>
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$w = \frac{50}{3}$ is a max. of the area $A$.

Dimensions that give maximum area: $w = \frac{50}{3} = 16 \frac{2}{3} \text{ m}$.

\[l = \frac{50 - 3 \cdot \frac{50}{3}}{2} = 25 \text{ m}.
\]

Maximum area $= wl = \frac{50}{3} \cdot 25 = \frac{1250}{3} = 416 \frac{2}{3} \text{ m}^2$. 

8. (10 points) The rate at which the amount of penicillin in a person's bloodstream decreases is proportional to the amount of penicillin, \( P(t) \) present at time \( t \). Suppose 50 mg of penicillin remains in the bloodstream 7 hours after an initial injection of 450 mg.

(a) Write a differential equation whose solution is \( P(t) \).

\[
P'(t) = kP(t)
\]

(b) Write the solution of the differential equation in part (a). Find the value(s) of any constant(s) that may appear in the solution.

\[
P(t) = Ae^{kt}
\]

\[
A = P(0) = 450
\]

\[
P(t) = 450e^{kt}
\]

When \( t = 7 \), \( P = 200 \) mg.

\[
50 = 450e^{7k}
\]

\[
e^{7k} = \frac{50}{450} = \frac{1}{9}
\]

\[
7k = \ln\left(\frac{1}{9}\right)
\]

\[
k = \frac{1}{7}\ln\left(\frac{1}{9}\right)
\]

So \( P(t) = 450e^{\frac{1}{7}\ln\left(\frac{1}{9}\right)t} \).

(c) At what time was 200 mg of penicillin present in the bloodstream? Include units in your answer.

\[
200 = 450e^{\frac{1}{7}\ln\left(\frac{1}{9}\right)t}
\]

\[
\frac{200}{450} = e^{\frac{1}{7}\ln\left(\frac{1}{9}\right)t}
\]

\[
\frac{200}{450} = e^{\frac{1}{7}\ln(\frac{4}{9})t}
\]

\[
\frac{1}{7}\ln\left(\frac{4}{9}\right)t = \ln\left(\frac{4}{9}\right)
\]

\[
t = \frac{7\ln\left(\frac{4}{9}\right)}{\ln\left(\frac{1}{9}\right)} \approx 2.58 \text{ hours}
\]

So about 2.58 hours after the initial injection, 200 mg of penicillin was present in the bloodstream.
9. (8 points) Suppose a function $f$ is such that $f(5) = 0$. The graph of $f'(x)$ is given below.

![Graph of f'(x)](image)

(a) Evaluate $\lim_{x\to 5} \frac{f(x)}{2x-10}$

$$
\lim_{x\to 5} \frac{f(x)}{2x-10} \xrightarrow{0} \lim_{x\to 5} \frac{f(x)}{2} = \frac{f'(5)}{2} = -\frac{6}{2} = -3.
$$

(Equation of $f'(x)$ is $y = -2x + 4$.

So $f'(5) = -2(5) + 4 = -6$)

(b) Let $h(x) = 3^x f(x)$. Is $h$ increasing at $x = 5$? Justify your answer.

$h'(x) = 3^x f'(x) + (ln 3) 3^x f(x)$

So $h'(5) = 3^5 f'(5) + (ln 3) 3^5 f(5)$ (if $5 = 0$)

So $h'(5) = 3^5(-6) < 0$.

So $h$ is decreasing at $x = 5$.

Hence $h$ is not increasing at $x = 5$. 