Math 105: Review for Exam II - Solutions

1. Find \( \frac{dy}{dx} \) for each of the following.
   
   (a) \( y = x^2 + 2x + e^x + e^{2x} + \ln 2 + \ln(2x) + \arctan 2 \)
   
   \[
   \frac{dy}{dx} = 2x + (\ln 2)2^x + 2e^{2x} + \frac{1}{2x} \cdot 2
   \]
   
   Note that \( e^x, \ln 2, \) and \( \arctan 2 \) are constants.

   (b) \( y = \sqrt{x} \cdot \arctan(5x) \)

   \[
   \frac{dy}{dx} = \frac{1}{2}x^{-1/2} \arctan(5x) + \sqrt{x} \cdot \frac{1}{1 + (5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1 + 25x^2}
   \]

   (c) \( y = \ln(\tan(2\cos(x^2))) \)

   \[
   \frac{dy}{dx} = \frac{1}{\tan(2\cos(x^2))} \cdot \sec^2(2\cos(x^2)) \cdot \ln(2\cos(x^2)) \cdot (-\sin(x^2)) \cdot 2x
   \]

   (d) \( y = \frac{x + e^x}{\cos 4 + \sin^5(6x)} \)

   Note that \( e^x \) and \( \cos 4 \) are constants.

   \[
   \frac{dy}{dx} = \frac{(1)(\cos 4 + \sin^5(6x)) - (x + e^x)(5\sin^4(6x) \cdot \cos(6x) \cdot 6)}{\cos 4 + \sin^5(6x))^2} 
   
   \text{Recall that } \sin^5(6x) = (\sin(6x))^5.
   \]

2. Consider the curve defined by \( x^3 + y^3 = \frac{9}{2}xy \) (known as the Folium of Descartes).

   (a) Find \( \frac{dy}{dx} \). Use implicit differentiation.

   \[
   3x^2 + 3y^2 \frac{dy}{dx} = \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx}
   
   3y^2 \frac{dy}{dx} - \frac{9}{2} \frac{dy}{dx} = \frac{9}{2}y - 3x^2
   
   \frac{dy}{dx} \left( 3y^2 - \frac{9}{2} \right) = \frac{9}{2}y - 3x^2
   
   \frac{dy}{dx} = \frac{9}{2}y - 3x^2
   
   \frac{dy}{dx} = \frac{9}{2}y - 3x^2
   
   \frac{dy}{dx} = 3y^2 - \frac{3}{x^2}
   \]

   (b) Verify that the point \((1,2)\) is on the curve above.

   We must check to see if the values \( x = 1 \) and \( y = 2 \) satisfy the equation above.

   \[
   x^3 + y^3 = \frac{9}{2}xy
   
   1^3 + 2^3 = \frac{9}{2} \cdot 1 \cdot 2
   
   9 \neq 9
   \]

   Thus, the point \((1,2)\) is on the curve.

   (c) Find the equation of the tangent line at the point \((1,2)\).

   We want \( y = mx + b \).

   \[
   m = \frac{\frac{9}{2} \cdot 2 - 3 \cdot 1^2}{3 \cdot 2^2 - \frac{3}{2} \cdot 1} = \frac{4}{5}, \text{ so } y = \frac{4}{5}x + b.
   \]

   Now plug in \( x = 1 \) and \( y = 2 \) to find \( b \).

   \[
   2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b
   \]

   Therefore, we have \( y = \frac{4}{5}x + \frac{6}{5} \).
3. Evaluate the following limits.
Throughout this solution, the symbol $\star$ will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form $0/0$; this may be "\text{indeterminate}" or $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $\frac{0}{\infty}$ or $\frac{\infty}{0}$ or $\frac{\infty}{\infty}$ or "has the form $\frac{0}{0}$" and so, by L'Hopital's Rule, is equal to $\text{something else}$.

The symbol $\triangle$ will serve the same purpose for the indeterminate forms $\infty/\infty$ and $-\infty/\infty$.

(a) \[ \lim_{x \to 1} \frac{x^3 - 1}{7 - 7x} \quad \star \quad \lim_{x \to 1} \frac{3x^2}{-7} = \frac{3}{-7} = \frac{3}{7}. \]

(b) \[ \lim_{x \to 0} \frac{1 - \cos(2x)}{3^x} = \frac{0}{1} = 0 \quad \text{Can't use (and don't need) L'Hopital's Rule!} \]

(c) \[ \lim_{x \to 0} \frac{1 - \cos(4x)}{5x^2} \quad \star \quad \lim_{x \to 0} \frac{4 \sin(4x)}{10x} \quad \star \quad \lim_{x \to 0} \frac{16 \cos(4x)}{10} = \frac{16}{10} = \frac{8}{5} \]

(d) \[ \lim_{x \to \infty} \frac{x^2}{2x} \quad \triangledown \quad \lim_{x \to \infty} \frac{2x}{2 \cdot 2x} \quad \triangledown \quad \lim_{x \to \infty} \frac{2}{2 - \ln 2} \cdot 2x = 0 \]

4. Rewrite $\tan(\arccos x)$ as an algebraic expression - no trigonometric or inverse trigonometric functions. [Students in the 8:00 section may omit this problem.]

Let $\theta = \arccos x$. That is, $\theta$ is the angle whose cosine is $x$.

\[ \tan(\arccos x) = \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{1 - x^2}}{x} \]

5. Consider the function $f(x) = x^4e^x$ with domain all real numbers.

(a) Find the $x$-value(s) of all roots ($x$-intercepts) of $f$.
The equation $x^4e^x = 0$ means $x^4 = 0$ (that is, $x = 0$) or $e^x = 0$ (no solution), so the only root is at $x = 0$.

(b) Find the $x$- and $y$-value(s) of all critical points and identify each as a local max, local min, or neither.

\[ f'(x) = 4x^3e^x + x^4e^x \]
\[ 0 = x^3e^x(4 + x) \]
\[ \Rightarrow x = 0, -4 \quad \text{Note that } e^x \text{ is never } 0. \]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -4$</td>
<td>positive</td>
</tr>
<tr>
<td>$-4 &lt; x &lt; 0$</td>
<td>negative</td>
</tr>
<tr>
<td>$4 &lt; x$</td>
<td>positive</td>
</tr>
</tbody>
</table>

$y$-values: $f(-4) = 256e^{-4} \approx 4.689$, $f(0) = 0$
So, $f$ has a local maximum at $(-4, 256e^{-4})$ and a local minimum at $(0, 0)$.

(c) Find the $x$- and $y$-value(s) of all global extrema and identify each as a global max or global min.
There is a global minimum at $(0, 0)$. There is no global maximum because as $x \to \infty$, $f(x) \to \infty$.
Note that as $x \to -\infty$, $f(x) \to 0$. You can verify this by using L'Hopital's Rule on $x^4/e^{-x}$. 

(d) Find the \( x \)-value(s) of all inflection points.

\[ f''(x) = 12x^2 e^x + 4x^3 e^x + 4x^3 e^x + x^4 e^x \]

Use Product Rule on each product in \( f'(x) \) above.

\[ 0 = e^x (x^4 + 8x^3 + 12x^2) \]
\[ 0 = e^x x^2 (x^2 + 8x + 12) \]
\[ 0 = e^x x^2 (x + 2)(x + 6) \]

\( \Rightarrow x = 0, -2, -6 \)

\[
\begin{array}{c|c|c|c|c}
 x < -6 & -6 < x < -2 & -2 < x < 0 & 0 < x \\
 f'' & \text{positive} & \text{negative} & \text{positive} & \text{positive} \\
 f & \text{concave up} & \text{concave down} & \text{concave up} & \text{concave up}
\end{array}
\]

So, the \( x \)-values of the inflection points of \( f \) are \( x = -2 \) and \( x = -6 \) but NOT \( x = 0 \).

(e) Sketch \( f \).

![Graph of f(x)](image)

6. How would your answers to the previous question change if the domain of \( f \) were \([-10, 10]\)?

There would be a global maximum at \((10, 10^4 e^{10})\). (And the graph would be restricted to \(-10 \leq x \leq 10\)).

7. You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is $288. If the glass for the sides costs $12 per square foot and the opaque material for the bottom costs $3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.

![Aquarium diagram](image)

**Goal:** Maximize volume

**Objective function:**

\[ V = x \cdot x \cdot y = x^2 y \]

We need to get this down to a function of just one variable, so we use the constraint equation:

\[
\begin{align*}
\text{total cost} &= \text{(cost of base)} + \text{(cost of two square ends)} + \text{(cost of two other sides)} \\
288 &= 3xy + 12 \cdot 2x^2 + 12 \cdot 2xy \\
288 &= 27xy + 24x^2 \\
288 - 24x^2 &= 27xy \\
\frac{288 - 24x^2}{27x} &= y
\end{align*}
\]

Substituting this back into the objective function gives
V = x^2y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27}(288x - 24x^3).

Now that we have V as a function of just one variable, we find its maximum.

\[ V'(x) = \frac{1}{27}(288 - 72x^2) \]
\[ 0 = \frac{1}{27}(288 - 72x^2) \]
\[ 0 = (288 - 72x^2) \]
\[ 72x^2 = 288 \]
\[ x^2 = \frac{288}{72} \]
\[ x = 2 \]

We discard \( x = -2 \) because lengths must be nonnegative.

Since \( V' \) is positive for \( x < 2 \) and negative for \( 2 < x \), we know that the maximum occurs at \( x = 2 \).

And \( y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9} \) so the dimensions are 2 by 2 by \( \frac{32}{9} \).

8. Use the Intermediate Value Theorem to explain why \( f(x) = x^3 - 4x^2 + 5 \) must have a root somewhere on the interval \([1, 2]\).

IVT: If \( f \) is continuous on \([a, b]\) and \( y \) is a number between \( f(a) \) and \( f(b) \), then there is a number \( c \) between \( a \) and \( b \) such that \( f(c) = y \).

Our function \( f \) is continuous on \([1, 2]\). We can compute that \( f(1) = 2 \) and \( f(2) = -3 \). Since 0 is a number between 2 and -3, the IVT says there is a number \( c \) between 1 and 2 such that \( f(c) = 0 \); this \( c \) is the desired root.

[In plainer English, \( f \) is positive at one endpoint and negative at the other. Since \( f \) is continuous, the only way its value can go from positive to negative is to go through zero; where \( f \) is zero is our root.]

9. Let \( y = \frac{x^3 \cos(x)}{x^2 + 1} \).

(a) Find \( \frac{dy}{dx} \) using the product and quotient rules.
\[
\frac{dy}{dx} = \frac{(3x^2 \cos(x) + (-\sin(x))x^3)(x^2 + 1) - 2x(x^3 \cos(x))}{(x^2 + 1)^2}
\]

(b) Find \( \frac{dy}{dx} \) using logarithmic differentiation. [Students in the 1:10 section may consider this as a bonus problem.]

\[
\ln y = \ln \left( \frac{x^3 \cos(x)}{x^2 + 1} \right) \quad \text{Take ln of each side.}
\]
\[
\ln y = 3 \ln x + \ln(\cos(x)) - \ln(x^2 + 1) \quad \text{Apply log rules.}
\]
\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \frac{1}{\cos(x)}(-\sin(x)) - \frac{1}{x^2 + 1}(2x) \quad \text{Differentiate each side.}
\]
\[
\frac{dy}{dx} = \left[ \frac{3 \sin(x)}{x} - \frac{2x}{\cos(x) \cdot x^2 + 1} \right] \cdot y
\]
\[
\frac{dy}{dx} = \left[ \frac{3 \sin(x)}{x} - \frac{2x}{x^2 + 1} \right] \cdot \frac{x^3 \cos(x)}{x^2 + 1}
\]

Do the two methods give the same answer? They should! Check for yourself.