1. Find $\frac{dy}{dx}$ for each of the following.

(a) $y = x^2 + 2x + e^x + 2e^x + \ln 2 + \ln (2x) + \arctan 2$

$$\frac{dy}{dx} = 2x + (\ln 2)2x + 2e^x + \frac{1}{2x} \cdot 2$$

Note that $e^x$, $\ln 2$, and $\arctan 2$ are constants.

(b) $y = \sqrt{x} \cdot \arctan (5x)$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \arctan(5x) + \sqrt{x} \cdot \frac{1}{1+(5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1+25x^2}$$

(c) $y = \ln(\tan(2 \cos(x^2)))$

$$\frac{dy}{dx} = \frac{1}{\tan(2\cos(x^2))} \cdot \sec^2(2\cos(x^2)) \cdot 2 \ln(\tan(2 \cos(x^2))) \cdot (-\sin(x^2)) \cdot 2x$$

(d) $y = \sin^5 \left( \frac{x + e^\pi}{\ln 4 + \arcsin 6x} \right)$

$$\frac{dy}{dx} = 5 \sin^4 \left( \frac{x + e^\pi}{\ln 4 + \arcsin 6x} \right) \cdot \cos \left( \frac{x + e^\pi}{\ln 4 + \arcsin 6x} \right) \cdot \frac{(1)(\ln 4 + \arcsin 6x) - (x + e^\pi) \left( \frac{1}{\sqrt{1-(6x)^2}} \cdot 6 \right)}{(\ln 4 + \arcsin 6x)^2}$$

Note that $\sin^5 w = (\sin w)^5$.

2. Consider the curve defined by $x^3 + y^3 = \frac{9}{2}xy$ (known as the Folium of Descartes).

(a) **Find $\frac{dy}{dx}$**. Use implicit differentiation.

$$3x^2 + 3y^2 \frac{dy}{dx} = \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx}$$

$$3y^2 \frac{dy}{dx} - \frac{9}{2}x \frac{dy}{dx} = \frac{9}{2}y - 3x^2$$

$$\frac{dy}{dx} \left( 3y^2 - \frac{9}{2}x \right) = \frac{9}{2}y - 3x^2$$

$$\frac{dy}{dx} = \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x}$$

(b) **Find the equation of the tangent line at the point (1,2)**.

We want $y = mx + b$.

$$m = \frac{\frac{9}{2} \cdot 2 - 3 \cdot 1^2}{3 \cdot 2^2 - \frac{9}{2} \cdot 1} = \frac{4}{5}, \text{ so } y = \frac{4}{5}x + b.$$ 

Now plug in $x = 1$ and $y = 2$ to find $b$.

$$2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b$$

Therefore, we have $y = \frac{4}{5}x + \frac{6}{5}$.

3. **Evaluate the following limits**.

Throughout this solution, the symbol ⋆ will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form 0/0; this may be "0/0" or $\frac{L'}{H'}$ or $\frac{H}{H}$ or = "0/0" or "has the form 0/0" and so, by L'Hopital’s Rule, is equal to" or something else. The symbol ♦ will serve the same purpose for the indeterminate form $\infty/\infty$.

(a) $\lim_{x \to 0} \frac{\sin 3x}{5x} \quad ⋆ \quad \lim_{x \to 0} \frac{3 \cos 3x}{5} = \frac{3}{5}$
5. Suppose the domain of \( f(x) \) is all reals and that \( f \) has an inverse function \( f^{-1}(x) \). Further, suppose that \( f(2) = 5 \) and \( f'(2) = e \). Finally, let \( h(x) = 1/f(x) \).

(a) What point must be on the graph of \( f^{-1}(x) \)?
Since \( f(2) = 5 \), we also know that \( f^{-1}(5) = 2 \); therefore, the point \((5, 2)\) is on the graph of \( f^{-1}(x) \).

(b) What point must be on the graph of \( h(x) \)?
Since \( f(2) = 5 \), we also know that \( h(2) = 1/5 \); therefore, the point \((2, 1/5)\) is on the graph of \( h(x) \). The moral of the story in parts (a) and (b) is that inverses and reciprocals are not the same.

(c) Give an example of a point that cannot be on the graph of \( f(x) \). Do not choose a point with \( x \)-value of 2.
Since \( f^{-1}(x) \) is a function, we know that for each of its inputs (such as 5) there is only one output (in this case, 2). This means that no point of the form \((5, k)\) can be on the graph of \( f^{-1}(x) \) (except for \( k = 2 \)). Therefore, no point of the form \((k, 5)\) can be on the graph of \( f(x) \) (except for \( k = 2 \)). [Note: this is the same as saying that \( f \) must be a one-to-one function; on \( f \), no two \( x \)-values correspond to the same \( y \)-value.]

(d) What is the value of the derivative of \( h(x) \) at \( x = 2 \)?
By the Quotient Rule (or Chain Rule if you prefer), \( h'(x) = \frac{0 \cdot f(x) - 1 \cdot f'(x)}{[f(x)]^2} = -\frac{f'(x)}{[f(x)]^2} = -\frac{e}{5^2} = -\frac{e}{25}. \)

5. Suppose that \( y = f(t) \) is a solution to the differential equation \( y' = \frac{1}{\pi} \arcsin t + y^2 \) and that \( f \left( \frac{\sqrt{2}}{2} \right) = \frac{1}{2} \). Find the equation of the tangent line to \( f \) at \( \left( \frac{\sqrt{2}}{2}, \frac{1}{2} \right) \).
We want \( y = mx + b \).

\[
m = \frac{1}{\pi} \arcsin \frac{\sqrt{2}}{2} + \left( \frac{1}{2} \right)^2 = \frac{1}{\pi} \cdot \frac{\pi}{4} + \frac{1}{4} = \frac{1}{2}, \text{ so } y = \frac{1}{2}x + b.
\]

Now plug in \( x = \frac{\sqrt{2}}{2} \) and \( y = \frac{1}{2} \) to find \( b \).

\[
\frac{1}{2} = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + b \Rightarrow \frac{1}{2} - \frac{\sqrt{2}}{4} = b
\]

Therefore, we have \( y = \frac{1}{2}x + \frac{1}{2} - \frac{\sqrt{2}}{4} \).
6. Simplify $\tan(\arccos x)$ as much as possible.

Let $\theta = \arccos x$. That is, $\theta$ is the angle whose cosine is $x$.

\[ x^2 + y^2 = 1^2 \Rightarrow y = \sqrt{1 - x^2} \]

\[ \tan(\arccos x) = \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sqrt{1 - x^2}}{x} \]

7. Consider the function $f(x) = x^4 e^x$ with domain all real numbers.

(a) Find the $x$-value(s) of all roots ($x$-intercepts) of $f$.

The equation $x^4 e^x = 0$ means $x^4 = 0$ (that is, $x = 0$) or $e^x = 0$ (no solution), so the only root is at $x = 0$.

(b) Find the $x$- and $y$-value(s) of all critical points and identify each as a local max, local min, or neither.

\[ f'(x) = 4x^3 e^x + x^4 e^x \]
\[ 0 = x^3 e^x (4 + x) \]
\[ \Rightarrow x = 0, -4 \]

Note that $e^x$ is never 0.

\[ \begin{array}{c|c|c|c}
\text{f'} & x < -4 & -4 < x < 0 & 4 < x \\
\hline
\text{f''} & \text{positive} & \text{negative} & \text{positive} \\
\end{array} \]

$y$-values: $f(-4) = 256e^{-4} \approx 4.689$, $f(0) = 0$

So, $f$ has a local maximum at $(-4, 256e^{-4})$ and a local minimum at $(0, 0)$.

(c) Find the $x$- and $y$-value(s) of all global extrema and identify each as a global max or global min.

There is a global minimum at $(0, 0)$. There is no global maximum because as $x \to \infty$, $f(x) \to \infty$. Note that as $x \to -\infty$, $f(x) \to 0$. You can verify this by using L'Hopital’s Rule on $x^4/e^{-x}$.

(d) Find the $x$-value(s) of all inflection points.

\[ f''(x) = 12x^2 e^x + 4x^3 e^x + 4x^3 e^x + x^4 e^x \]

Use Product Rule on each product in $f'(x)$ above.

\[ 0 = e^x(x^4 + 8x^3 + 12x^2) \]
\[ 0 = e^x x^2(x^2 + 8x + 12) \]
\[ 0 = e^x x^2(x + 2)(x + 6) \]
\[ \Rightarrow x = 0, -2, -6 \]

\[ \begin{array}{c|c|c|c|c}
\text{f''} & x < -6 & -6 < x < -2 & -2 < x < 0 & 0 < x \\
\hline
\text{f'} & \text{positive} & \text{negative} & \text{positive} & \text{positive} \\
\end{array} \]

$y$-values: $f(-6) = 256e^{-6} \approx 0.0026$, $f(-2) = 0.135$, $f(0) = 0$

So, the $x$-values of the inflection points of $f$ are $x = -2$ and $x = -6$ but NOT $x = 0$. 
8. How would your answers to the previous question change if the domain of $f$ were $[-10, 10]$? There would be a global maximum at $(10, 10^{4.10})$. (And the graph would be restricted to $-10 \leq x \leq 10$).

9. Circle *always*, *sometimes*, or *never* to make each statement below correct.

(a) If $f'(1) = 0$ then $f$ *always*/*sometimes*/never has a critical point at $x = 1$.
   A critical point is where $f'$ is 0 or undefined.

(b) If $f'(2) = 0$ then $f$ *always*/*sometimes*/never has a local maximum or local minimum at $x = 2$.
   $f$ might instead have a terrace point at $x = 2$; we need $f'$ to change sign at $x = 2$ in order to guarantee a local extremum there.

(c) If $x = 3$ is a critical point of $f$, then $f'(3)$ is *always*/*sometimes*/never 0.
   It may also be that $f'(3)$ is undefined.

(d) If $f''(4) = 0$, then $f$ *always*/*sometimes*/never has an inflection point at $x = 4$.
   We need $f''$ to change sign at $x = 4$ to guarantee an inflection point there.
   For example, if $f(x) = (x - 4)^4$ then $f''(4) = 0$ but $f$ has a local minimum rather than an inflection point at $x = 4$.
   However, if $f(x) = (x - 4)^3$ then $f''(4) = 0$ and $f$ does have an inflection point at $x = 4$.
   Also see what happens at $x = 0$ in problem 7(d).

(e) If $f$ has a global maximum at $x = 5$, then $f'(5)$ is *always*/*sometimes*/never 0.
   $f'(5)$ might also be undefined, or $x = 5$ might be an endpoint of the domain.

(f) If $f'(6) = 0$ and $f''(6) = -2$, then $f$ *always*/*sometimes*/never has a local maximum at $x = 6$.
   If $f$ is concave down with a horizontal slope at $x = 6$, then $f$ must have a local maximum there.

(g) If $f'(7) = 0$ and $f''(7) = 0$, then $f$ *always*/*sometimes*/never has a local extremum at $x = 7$.
   This means that the second derivative test is inconclusive, so you need to use a different test.
   For example, if $f(x) = (x - 7)^4$ then $f'(7) = 0$ and $f''(7) = 0$ and $f$ has a local minimum at $x = 7$.
   However, if $f(x) = (x - 7)^3$ then $f'(7) = 0$ and $f''(7) = 0$ and $f$ has an inflection point but not a local extremum at $x = 7$. 
10. The rate of change of a population $P(t)$ of eels is proportional to the size of the population. When the population is 40000, it is growing at a rate of 400 eels per year. At time $t = 0$, the population is 10000.

(a) Write a differential equation whose solution is $P(t)$.
Rate of change ($P'$) is (=) proportional to ($k$) size of population ($P$) means $P' = kP$.
What’s the value of $k$? When $P = 40000$, we know $P' = 400$. That is, $400 = k \cdot 40000$, so $k = .01$. Thus, we have $P' = .01P$.

(b) Solve your differential equation.
The general solution is $P(t) = Ae^{.01t}$.
What’s the value of $A$? When $t = 0$, we know $P = 10000$. That is, $10000 = Ae^0 = A$, so $A = 10000$.
Thus, we have $P(t) = 10000e^{.01t}$.

(c) When will the population reach 60000?

$$60000 = 10000e^{.01t}$$
$$6 = e^{.01t}$$
$$\ln 6 = \ln e^{.01t}$$
$$\ln 6 = .01t$$
Take ln of each side.
Recall that $\ln e^z = z$.
$$t = 100 \ln 6 \approx 179.176 \text{ years}$$

11. For the following questions, suppose that the graph shown is $f'(x)$ and that $f(2) = 3$.

(a) If $g(x) = f(x) \ln x$, is $g$ increasing, decreasing or stationary at $x = 2$?
By the Product Rule, we know $g'(x) = f'(x) \ln x + f(x) \cdot \frac{1}{x}$.

Therefore, we have $g'(2) = f'(2) \ln 2 + f(2) \cdot \frac{1}{2}$
$$= 0 \cdot \ln 2 + 3 \cdot \frac{1}{2}$$
$$= \frac{3}{2}$$

Since $g'(2)$ is positive, we know that $g$ is increasing at $x = 2$.

(b) If $h(x) = f(x^2)$, is $h$ increasing, decreasing or stationary at $x = -2$?
By the Chain Rule, we know $h'(x) = f'(x^2) \cdot 2x$.

Therefore, we have $h'(-2) = f'((-2)^2) \cdot 2(-2)$
$$= f'(4) \cdot (-4)$$
$$= \text{a positive number} \cdot (-4)$$
$$= \text{a negative number}$$
[Note that from the graph above, we see that \( f'(4) \) is positive (above the x-axis).]

Since \( h'(-2) \) is negative, we know \( h \) is decreasing at \( x = -2 \).

(c) If \( h(x) = f(x^2) \), is \( h \) concave up, concave down or neither at \( x = -\sqrt{2} \)?

We found \( h'(x) \) in the previous problem. We now use the Product and Chain Rules to find \( h''(x) \).

\[
h''(x) = \left[ f''(x^2) \cdot 2x \right] \cdot 2x + f'(x^2) \cdot 2
\]

\[
h''(-\sqrt{2}) = f''((-\sqrt{2})^2) \cdot 4(-\sqrt{2})^2 + f'((-\sqrt{2})^2) \cdot 2
\]

\[
= f''(2) \cdot 8 + f'(2) \cdot 2
\]

\[
= \text{a positive number} \cdot 8 + 0 \cdot 2
\]

\[
= \text{a positive number}
\]

[Note that since the graph of \( f' \) has a positive slope at \( x = 2 \), \( f''(2) \) is positive.]

Since \( h''(-\sqrt{2}) \) is positive, we know \( h \) is concave up at \( x = -\sqrt{2} \).

12. You are designing an 18 ft\(^3\) box that will have a square bottom and no top. The material for the bottom costs 40 cents per square foot and the material for the sides costs 30 cents per square foot. What dimensions give the least total cost? Be sure to show how you know you have found the minimum.

![Diagram of box with dimensions x and y]

Goal: minimize cost

**Objective function:** \( C = 40 \cdot x^2 + 30 \cdot 4xy \)

We need to get this down to a function of just one variable, so we use the

**constraint equation:** volume = 18 = \( x^2y \).

Solving for \( y \), we have \( y = \frac{18}{x^2} \).

Substituting this back into the objective function gives

\[
C = 40 \cdot x^2 + 30 \cdot 4x \cdot \frac{18}{x^2} = 40x^2 + \frac{2160}{x}.
\]

Now that we have \( C \) as a function of just one variable, we find its minimum.

\[
C'(x) = 80x - \frac{2160}{x^2}
\]

\[
0 = 80x - \frac{2160}{x^2}
\]

\[
\frac{2160}{x^2} = 80x
\]

\[
\frac{2160}{80} = x^3
\]

\[
3 = x
\]

Since \( C' \) is negative for \( 0 < x < 3 \) and positive for \( 3 < x \), we know that the minimum occurs at \( x = 3 \).

And \( y = \frac{18}{x^2} = \frac{18}{3^2} = 2 \), so the dimensions are 3 by 3 by 2.
13. Use the Intermediate Value Theorem to explain why \( f(x) = x^3 - 4x^2 + 5 \) must have a root somewhere on the interval \([1, 2]\).

IVT: If \( f \) is continuous on \([a, b]\) and \( y \) is a number between \( f(a) \) and \( f(b) \), then there is a number \( c \) between \( a \) and \( b \) such that \( f(c) = y \).

Our function \( f \) is continuous on \([1, 2]\). We can compute that \( f(1) = 2 \) and \( f(2) = -3 \). Since 0 is a number between 2 and \(-3\), the IVT says there is a number \( c \) between 1 and 2 such that \( f(c) = 0 \); this \( c \) is the desired root.

[In plainer English, \( f \) is positive at one endpoint and negative at the other. Since \( f \) is continuous, the only way its value can go from positive to negative is to go through zero; where \( f \) is zero is our root.]