NAME:

Instruction: Read each question carefully. Explain ALL your work and give reasons to support your answers.
Advice: DON’T spend too much time on a single problem.

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1. Evaluate each of the following indefinite integrals (be sure to indicate what techniques you use).

(10 pts.) (a) 
\[ \int \cos^2 x \sin x \, dx. \]

Let \( u = \cos x \) so that \( du = -\sin x \, dx \). It follows that
\[
\int \cos^2 x \sin x \, dx = -\int u^2 \, du = -\frac{u^3}{3} + C \\
= -\frac{\cos^3 x}{3} + C.
\]

(10 pts.) (b) 
\[ \int x \cos(3x) \, dx. \]

Let \( u = x \) and \( dv = \cos(3x) \, dx \). Then, \( du = dx \) and \( v = \frac{\sin(3x)}{3} \). It follows from the technique of integration by parts that
\[
\int x \cos(3x) \, dx = \frac{x \sin(3x)}{3} - \int \frac{1}{3} \sin(3x) \, dx \\
= \frac{x}{3} \sin(3x) + \frac{1}{9} \cos(3x) + C.
\]
2. Evaluate each of the following indefinite integrals (be sure to indicate what techniques you use).

(10 pts.) (a) 
\[ \int \frac{x^3 + 1}{x^2 - 4} \, dx. \]

Since the numerator has higher degree than that of the denominator, we use long division to write the rational function as 
\[ \frac{x^3 + 1}{x^2 - 4} = x + \frac{4x + 1}{(x + 2)(x - 2)}. \]
Write \( \frac{4x + 1}{(x + 2)(x - 2)} = \frac{A}{x + 2} + \frac{B}{x - 2} \) so that 
\[ 4x + 1 \equiv A(x - 2) + B(x + 2). \]
At \( x = 2 \), we have \( 9 = 4B \) or \( B = \frac{9}{4} \). At \( x = -2 \), we have \( -7 = -4A \) or \( A = \frac{7}{4} \). Thus,
\[ \frac{x^3 + 1}{x^2 - 4} = x + \frac{7}{4} \cdot \frac{1}{x + 2} + \frac{9}{4} \cdot \frac{1}{x - 2}. \]
Hence,
\[ \int \frac{x^3 + 1}{x^2 - 4} \, dx = \frac{x^2}{2} + \frac{7}{4} \ln |x + 2| + \frac{9}{4} \ln |x - 2| + C. \]

(10 pts.) (b) 
\[ \int \frac{1}{(x^2 + 1)^{3/2}} \, dx. \]

Let \( x = \tan \theta \) so that \( dx = \sec^2 \theta \, d\theta \). It follows that
\[ \int \frac{1}{(x^2 + 1)^{3/2}} \, dx = \int \frac{\sec^2 \theta}{(\tan^2 \theta - 1)^{3/2}} \, d\theta \]
\[ = \int \frac{1}{\sec \theta} \, d\theta = \int \cos \theta \, d\theta \]
\[ = \sin \theta + C \quad \text{(now use the triangle of substitution)} \]
\[ = \frac{x}{\sqrt{1 + x^2}} + C. \]
3. Evaluate each of the following improper integrals.

(10 pts.) (a) \[ \int_{0}^{\infty} \frac{x}{(x^2 + 1)^2} \, dx \]

First, \( \int_{0}^{\infty} \frac{x}{(x^2 + 1)^2} \, dx = \lim_{b \to \infty} \int_{0}^{b} \frac{x}{(x^2 + 1)^2} \, dx \). Now, let \( u = x^2 + 1 \) so \( du = 2x \, dx \). It follows that
\[ \int \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2} \int \frac{du}{u^2} = -\frac{1}{2u} + C \]
\[ = -\frac{1}{2(x^2 + 1)} + C. \]

Now,
\[ \int_{0}^{\infty} \frac{x}{(x^2 + 1)^2} \, dx = \lim_{b \to \infty} \int_{0}^{b} \frac{x}{(x^2 + 1)^2} \, dx \]
\[ = \lim_{b \to \infty} \left( -\frac{1}{2(x^2 + 1)} + C \right) \]
\[ = \frac{1}{2}. \]

(10 pts.) (b) \[ \int_{0}^{2} \frac{dx}{(x-1)^{2/3}} \]

First note that this integral is improper since \( \frac{1}{(x-1)^{2/3}} \) is not defined at \( x = 1 \). Thus, we rewrite the integral as
\[ \int_{0}^{2} \frac{dx}{(x-1)^{2/3}} = \int_{0}^{1} \frac{dx}{(x-1)^{2/3}} + \int_{1}^{2} \frac{dx}{(x-1)^{2/3}} \]
\[ = \lim_{b \to 1} \int_{0}^{b} \frac{dx}{(x-1)^{2/3}} + \lim_{c \to 1} \int_{c}^{2} \frac{dx}{(x-1)^{2/3}}. \]

A simple substitution with \( u = x - 1 \) shows that \( \int \frac{dx}{(x-1)^{2/3}} = \int u^{-2/3} \, du = 3u^{1/3} + C = 3(x - 1)^{1/3} + C \). Now,
\[ \lim_{b \to 1} \int_{0}^{b} \frac{dx}{(x-1)^{2/3}} = \lim_{b \to 1} (3(b - 1)^{1/3} - 3(-1)^{1/3}) = 3 \]
and
\[ \lim_{c \to 1} \int_{c}^{2} \frac{dx}{(x-1)^{2/3}} = \lim_{c \to 1} (3(1)^{1/3} - 3(c - 1)^{1/3}) = 3. \]

Therefore,
\[ \int_{0}^{2} \frac{dx}{(x-1)^{2/3}} = 3 + 3 = 6. \]
4. Let \( f(x) = \frac{1}{2x-1} \).

(8 pts.) (a) Find the third-order Taylor polynomial \( P_3(x) \) of \( f(x) \) centered at \( x_0 = 1 \).

Since \( f(x) = (2x - 1)^{-1} \), we have \( f'(x) = -2(2x - 1)^{-2} \), \( f''(x) = 8(2x - 1)^{-3} \) and \( f'''(x) = -48(2x-1)^{-4} \). It follows that \( f(1) = 1 \), \( f'(1) = -2 \), \( f''(1) = 8 \) and \( f'''(1) = -48 \). Now, the third-order Taylor polynomial is given by

\[
P_3(x) = 1 + (-2)(x - 1) + \frac{(8)}{2!}(x - 1)^2 + \frac{(-48)}{3!}(x - 1)^3
= 1 - 2(x - 1) + 4(x - 1)^2 - 8(x - 1)^3.
\]

(6 pts.) (b) Find the third-order Maclaurin polynomial \( M_3(x) \) of \( f(x) \).

Using (a), with \( x_0 = 0 \), we have \( f(0) = -1 \), \( f'(0) = -2 \), \( f''(0) = -8 \), \( f'''(0) = -48 \). Thus the third-order Maclaurin polynomial is given by

\[
M_3(x) = (-1) + (-2)x + \frac{(8)}{2!}x^2 + \frac{(-48)}{3!}x^3
= -1 - 2x - 4x^2 - 8x^3.
\]

(6 pts.) (c) What is the maximum error committed by using \( M_3(x) \) (as in part (b)) over the interval \([1, 2]\), according to Taylor’s Theorem? [Hint: how do you obtain \( K_4 \)?]

To obtain \( K_4 \), first note that \( f^{(4)}(x) = 384(2x - 1)^{-5} \). Over the interval \([1, 2]\), \( |f^{(4)}(x)| \leq 385(2 - 1)^{-5} = 384 \). Thus, choose \( K_4 = 384 \). According to Taylor’s theorem, we have

\[
|f(x) - M_3(x)| \leq \frac{K_4}{4!}|x - 0|^4 \leq \frac{384}{4!} \cdot 2^4 = 256.
\]
5. (12 pts.) (a) Use comparison to determine whether the following improper integral converges or diverges. Justify your answer.

\[ \int_{4}^{\infty} \frac{1}{x^2 \ln x} \, dx \]

For \( x \geq 4 \), \( \ln x > 1 \) so \( x^2 \ln x > x^2 \). It follows that \( \frac{1}{x^2 \ln x} < \frac{1}{x^2} \) so that

\[ 0 < \int_{4}^{\infty} \frac{1}{x^2 \ln x} \, dx < \int_{4}^{\infty} \frac{1}{x^2} \, dx < \int_{1}^{\infty} \frac{dx}{x^2} \]

which converges by the \( p \)-test with \( p = 2 > 1 \). Thus we conclude that \( \int_{4}^{\infty} \frac{1}{x^2 \ln x} \, dx \) converges.

(8 pts.)(b) Consider the following function

\[ f(x) = \begin{cases} \frac{C}{(x+1)^3}, & \text{for } x \geq 0; \\ 0, & \text{otherwise}. \end{cases} \]

For what value of \( C \) is \( f(x) \) a probability density function?

For \( f \) to be a p.d.f., \( f(x) \geq 0 \) and \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \). So \( C \geq 0 \). Note that

\[ \int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{\infty} \frac{C}{(x+1)^3} \, dx = \lim_{b \to \infty} \int_{0}^{b} \frac{C}{(x+1)^3} \, dx \]

\[ = \lim_{b \to \infty} -\frac{C}{2} (x+1)^{-2} \bigg|_{0}^{b} = -\frac{C}{2} \lim_{b \to \infty} [(b+1)^{-2} - (1)^{-2}] \]

\[ = -\frac{C}{2} [0 - 1] = \frac{C}{2}. \]

The condition \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \) implies that \( \frac{C}{2} = 1 \) or \( C = 2 \).