1. Suppose that $V$ is a vector space and $H$ is a subspace of $V$ and the vectors $v_1$, $v_2$, $\ldots$, $v_j$ all belong to $H$. Give the definition of what it means to say that the set $\{v_1, v_2, \ldots, v_j\}$ is a basis of $H$.

2a. Given an $m \times n$ matrix $A$, the Null Space of $A$ (i.e., $\text{Nul}(A)$) is defined as the set of solutions of what matrix equation?

2b. In the answer to 2a, is this set a subspace of $\mathbb{R}^q$ for $q = m$? or $q = n$?

2c. Let $A = \begin{bmatrix} 4 & 7 & 1 & 5 \\ 6 & 10 & 4 & 6 \\ 5 & 8 & 5 & 4 \end{bmatrix}$. Find a basis for the Null Space of $A$. Show any rref’d matrices involved. Write your final answer in correct basis notation.
3. Suppose that $A$ is a $3 \times 3$ matrix. Suppose the following sequence of elementary row operations, labeled Op1, Op2 and Op3, turn $A$ into the identity matrix $I_3$.
   Op1: Rows one and two of $A$ are swapped.
   Op2: In the matrix that results from $A$ after Op1 is finished, 5 copies of row one are subtracted from row three.
   Op3: In the matrix resulting after Op2 is completed, 3 copies of row two are added to row one.

3a. Let $P$, $M$ and $W$ be the three elementary matrices which represent Op1, Op2 and Op3, respectively. Explicitly find $P$, $M$ and $W$ and label which is which.

3b. Find $A^{-1}$, and explain how you found it.

3c. Find $A$. Hint: $(A^{-1})^{-1} = A$. There are several ways available to you for finding this expression.

3d. Find $(A^{-1})^T$. 
4a. Let \( K = \begin{bmatrix} 6 & 2 & 5 & 2 \\ 3 & 1 & 3 & 3 \\ 9 & 3 & 8 & 5 \end{bmatrix} \); label the column vectors of \( K \) as \( k_1, k_2, k_3, \) and \( k_4 \), respectively. Let \( M = \begin{bmatrix} 2 & 5 & 2 & 6 \\ 3 & 3 & 1 & 3 \\ 5 & 8 & 3 & 9 \end{bmatrix} \), so \( M \) has the same columns as \( K \), just in a different order. Explain why the column spaces of \( K \) and \( M \) are the same:

4b. Theorem 6 in section 4.3 tells you that which column vectors of a matrix \( A \) form a basis of the Column Space of \( A \)?

Answer to 4b:

Now, here are two facts that you will find useful:

\[
\text{rref}( [K|I_3] ) = \begin{bmatrix} 1 & 1/3 & 0 & -3 & 0 & -8/3 & 1 \\ 0 & 0 & 1 & 4 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \quad \text{rref}( [M|I_3] ) = \begin{bmatrix} 1 & 0 & -1/9 & -1/3 & 0 & 8/9 & -1/3 \\ 0 & 1 & 4/9 & 4/3 & 0 & -5/9 & 1/3 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}
\]

4c. Use the above facts and the answer to (4b) to find a basis for \( \text{Col}(K) \). Use correct notation for writing a basis.

4d. Now use the facts to get a basis for \( \text{Col}(M) \). (Since the column spaces \( \text{Col}(K) \) and \( \text{Col}(M) \) are the same, you now have two different bases for the same subspace).

4d. If you add the column vectors of \( K \) you get \( \begin{bmatrix} 15 \\ 10 \\ 25 \end{bmatrix} \). This vector is in both column spaces of course. Show how to write it as a LC of the basis vectors you found in (4c). Then show how to write it as a LC of the basis vectors in (4d). Show any rref’s you use.

4 BONUS. Through use of super-augmented matrices, the two facts give another way to show that \( \text{Col}(K) \) and \( \text{Col}(M) \) are the same. Explain what it is.
5. Let $H$ be the subset of all matrices in $M_{2 \times 2}$ of the form \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\] for which $5a + 2c = 0$ and $3a + 4b = 0$.

5A: Give an example of a member of $H$ in which all the entries are different and none of them are 0.

5B: Here’s a fact: $H$ is a subspace of $M_{2 \times 2}$. Write a correct proof of this fact in the style developed in class. You have three things to show.
6A. Define $T : \mathbb{P}_3 \to \mathbb{P}_2$ by $T(ax^3 + bx^2 + cx + d) = (4a + 5b)x^2 + 6c$. Prove “condition ONE” of the definition of linear transformation holds for $T$ or give a counterexample that shows it does not.

6B: Is $T$ a one-to-one LT? Either prove $T$ is one-to-one, or explain using a counterexample that it is not one-to-one.