INTEGRATION TIPS

- Substitution: usually let \( u \) = a function that’s “inside” another function, especially if \( du \) (possibly off by a multiplying constant) is also present in the integrand.

- Parts: \( \int u \, dv = uv - \int v \, du \) or \( \int uv' \, dx = uv - \int u'v \, dx \)

How to choose which part is \( u \)? Let \( u \) be the part that is higher up in the LIATE mnemonic below. (The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter A is replaced by P, which stands for “polynomial.”)

Logarithms (such as \( \ln x \))
Inverse trig (such as \( \arctan x \), \( \arcsin x \))
Algebraic (such as \( x \), \( x^2 \), \( x^3 + 4 \))
Trig (such as \( \sin x \), \( \cos 2x \))
Exponentials (such as \( e^x \), \( e^{3x} \))

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

Partial Fractions: here’s an illustrative example of the setup.

\[
\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}
\]

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor \( (x - 3) \) on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term \( (Dx + E) \) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>Square Root of Positive Term</th>
<th>Square Root of Negative Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = a \sin t )</td>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( \sqrt{a^2 + x^2} )</td>
</tr>
<tr>
<td>( x = a \tan t )</td>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( \sqrt{x^2 + a^2} )</td>
</tr>
<tr>
<td>( x = a \sec t )</td>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( \sqrt{x^2 + a^2} )</td>
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</tbody>
</table>

\[
\sin^2 x + \cos^2 x = 1 \\
\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} \\
\tan^2 x + 1 = \sec^2 x \\
\cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2} \\
\sin(2x) = 2\sin x \cos x
\]

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>( \int \sin^m x \cos^n x , dx )</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
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<tr>
<td>( m ) odd</td>
<td>Break off one factor of ( \sin x ) and substitute ( u = \cos x ).</td>
<td>( \sin^2 x = 1 - \cos^2 x )</td>
</tr>
<tr>
<td>( n ) odd</td>
<td>Break off one factor of ( \cos x ) and substitute ( u = \sin x ).</td>
<td>( \cos^2 x = 1 - \sin^2 x )</td>
</tr>
<tr>
<td>( m ) even AND ( n ) even</td>
<td>Use ( \sin^2 x + \cos^2 x = 1 ) to reduce to only powers of ( \sin x ) or only powers of ( \cos x ), then use table of integrals #39–42 or identities shown to right of this box.</td>
<td>( \sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} ) or ( \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2} ).</td>
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\int \tan^m x \sec^n x \, dx
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<td>Break off one factor of ( \sec x \tan x ) and substitute ( u = \sec x ).</td>
</tr>
<tr>
<td>( n ) even</td>
<td>Break off one factor of ( \sec^2 x ) and substitute ( u = \tan x ).</td>
</tr>
<tr>
<td>( m ) even AND ( n ) odd</td>
<td>Use identity at right to reduce to powers of ( \sec x ) alone.</td>
</tr>
<tr>
<td></td>
<td>Then use table of integrals #51 or integration by parts.</td>
</tr>
</tbody>
</table>

Useful Trigonometric Derivatives and Antiderivatives

\[
\frac{d}{dx} \tan x = \sec^2 x \\
\frac{d}{dx} \sec x = \sec x \tan x \\
\int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

- Improper integrals: look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

\[
\lim_{x \to \infty} e^x = \infty \\
\lim_{x \to \infty} e^{-x} = 0 \\
\lim_{x \to \infty} 1/x = 0 \\
\lim_{x \to \infty} 1/x = \infty \\
\lim_{x \to \infty} \ln x = \infty \\
\lim_{x \to 0^+} \ln x = -\infty
\]

1. Evaluate the following.

(a) Let \( u = \sin x \), so \( du = \cos x \, dx \).

\[
\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x(1 - \sin^2 x) \cos x \, dx
= \int u^6(1 - u^2) \, du
= \int (u^6 - u^8) \, du
= \frac{u^7}{7} - \frac{u^9}{9} + C
= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C
\]

(b) Let \( x = 10 \tan t \), so \( dx = 10 \sec^2 t \, dt \).

\[
x^2 + 10^2 = y^2 \Rightarrow y = \sqrt{x^2 + 100} \\
\sec t = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2 + 100}}{10} \\
\tan t = \frac{\text{opp}}{\text{adj}} = \frac{x}{10}
\]
\[
\int \frac{dx}{\sqrt{100 + x^2}} = \int \frac{10 \sec^2 t \, dt}{\sqrt{100 + 100 \tan^2 t}} \\
= \int \frac{10 \sec^2 t \, dt}{10 \sqrt{1 + \tan^2 t}} \\
= \int \frac{\sec^2 t \, dt}{\sqrt{\sec^2 t}} \\
= \int \sec t \, dt \\
= \ln |\sec t + \tan t| + C
\]

Now use the identity \(1 + \tan^2 t = \sec^2 t\).

\[
= \ln \left|\frac{\sqrt{x^2 + 100}}{10} + \frac{x}{10}\right| + C
\]

Now use the triangle above.

(c) This is an improper integral, so we need to use a limit.

\[
\int_3^\infty \frac{dx}{x(\ln x)^{100}} = \lim_{t \to \infty} \int_3^t \frac{1}{x(\ln x)^{100}} \, dx \\
= \lim_{t \to \infty} \int_{x=3}^{x=t} \frac{1}{u^{100}} \, du \\
= \lim_{t \to \infty} \left[ \frac{u^{-99}}{-99} \right]_{x=3}^{x=t} \\
= \lim_{t \to \infty} \left[ -\frac{1}{99(\ln x)^{99}} \right]_3 \\
= 0 - \left( -\frac{1}{99(\ln 3)^{99}} \right) \\
= \frac{1}{99(\ln 3)^{99}} \\
\]

So, the integral converges (to this value).

(d) We'll use integration by parts: \(u = x \Rightarrow du = dx\) and \(dv = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2}\).

\[
\int_0^\infty xe^{-2x} \, dx = \lim_{t \to \infty} \int_0^t xe^{-2x} \, dx \\
= \lim_{t \to \infty} \left[ \frac{x e^{-2x}}{-2} \right]_0^t - \int_0^t e^{-2x} \, dx \\
= \lim_{t \to \infty} \left[ \frac{e^{-2x}}{-2} - \frac{e^{-2x}}{4} \right]_0^t \\
= \lim_{t \to \infty} \left[ -\frac{1}{2e^{2t}} - \frac{1}{4e^{2t}} \right]_0^t \\
= \lim_{t \to \infty} \left[ \frac{-t}{2e^{2t}} - \frac{1}{4e^{2t}} - \left( \frac{0}{4e^0} - \frac{1}{4e^0} \right) \right] \\
= (0 - 0) - (0 - 1/4) \\
= 1/4
\]

So, the integral converges (to this value).

(e) Partial Fractions:

Write \(\frac{3x^2 + 2x - 13}{(x-3)(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 3}\). Now multiply both sides by \((x-3)(x^2 + 1)\) to get
\[3x^2 + 2x - 13 = (Ax + B)(x - 3) + C(x^2 + 1)\].

Let \(x = 3\). Then \(20 = C(10)\), so \(C = 2\).

Let \(x = 0\). Then \(-13 = B(-3) + 2(1)\), so \(B = 5\).

Let \(x = 1\). Then \(-8 = (A(1) + 5)(-2) + 2(2)\), so \(A = 1\).

\[
\int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx = \int \left[ \frac{x + 5}{x^2 + 1} + \frac{2}{x - 3} \right] dx
\]

Let \(u = x^2 + 1\), so \(du = 2xdx\).

\[
\ln \left( \frac{x^2 + 1}{2} \right) + 5 \arctan x + 2 \ln |x - 3| + D
\]

(f) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

\[
\frac{4x^2 - 3x + 2}{x - 6} = \frac{4x^3 - 27x^2 + 20x - 17}{4x^3 - 24x^2 - 3x^2 + 18x} - \frac{2x}{6}
\]

Now, we compute the integral.

\[
\int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx = \int \left[ 4x^2 - 3x + 2 - \frac{5}{x - 6} \right] dx = \frac{4x^3}{3} - \frac{3x^2}{2} + 2x - 5 \ln |x - 6| + C
\]

(g) This integral is improper at \(x = 1\) because the integrand has a vertical asymptote there, so we split into two integrals.

\[
\int_{-1}^{5} \frac{1}{(x - 1)^6} \, dx = \int_{-1}^{1} \frac{dx}{(x - 1)^6} + \int_{1}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to -1^-} \int_{-1}^{a} \frac{dx}{(x - 1)^6} + \lim_{b \to 1^+} \int_{b}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to -1^-} \left[ \frac{-1}{5(a - 1)^5} \right]_{-1}^{a} + \lim_{b \to 1^+} \left[ \frac{-1}{5(b - 1)^5} \right]_{b}^{5}
\]

Since \(\lim_{a \to -1^-} \frac{-1}{5(a - 1)^5} = \infty\) and \(\lim_{b \to 1^+} \frac{-1}{5(b - 1)^5} = \infty\), this integral diverges (to \(\infty\)).
2. Find the second-order Taylor polynomial for \( f(x) = \sqrt{x} \) based at \( x_0 = 100 \). Then use your polynomial to estimate \( \sqrt{105} \).

\[
\begin{align*}
f(x) &= x^{1/2} \\
f'(x) &= \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \\
f''(x) &= -\frac{1}{4} x^{-3/2} = -\frac{1}{4x^{3/2}}
\end{align*}
\]

\[
P_2(x) = f(100) + f'(100)(x - 100) + \frac{f''(100)}{2!} (x - 100)^2
\]

\[
= 10 + \frac{x - 100}{20} - \frac{(x - 100)^2}{8000}
\]

Now, \( \sqrt{105} \approx P_2(105) = 10 + \frac{105 - 100}{20} - \frac{(105 - 100)^2}{8000} = \frac{3279}{320} \).

3. What is the largest possible error that could have occurred in your estimate of \( \sqrt{105} \)?

We know that \(|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!} |x - x_0|^{n+1} \).

In this case, \( n = 2 \), \( x_0 = 100 \), and \( x = 105 \).

\[
K_3 = \max_{[100, 105]} |f'''(x)| = \max_{[100, 105]} \left| \frac{3}{8x^{5/2}} \right| = \frac{3}{8 \cdot 100^{5/2}} = \frac{3}{800,000}
\]

Putting this all together, we have \(|f(x) - P_2(x)| \leq \frac{3}{3!} (105 - 100)^3 = \frac{1}{12800} \).

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) \( \int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx \)

For all \( x \geq 1 \), we have \( \frac{6 + \cos x}{x^{0.99}} \geq \frac{6 - 1}{x^{0.99}} = \frac{5}{x^{0.99}} \) because the minimum value of \( \cos x \) is \(-1\).

Since \( \int_1^\infty \frac{5}{x^{0.99}} \) diverges (compute yourself or notice that \( p = 0.99 < 1 \)), we know that the integral in question must diverge too.

(b) \( \int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \)

For all \( x \geq 1 \), we have \( \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^5} = \frac{4}{x^2} \). (We’ve made the denominator smaller and the numerator larger, so the new fraction is larger.)

\[
4 \int_1^\infty \frac{dx}{x^2} = 4 \lim_{t \to \infty} \int_1^t \frac{dx}{x^2}
\]

\[
= 4 \lim_{t \to \infty} \left[ \frac{1}{1} \right]^{1/t}
\]

\[
= 4 \lim_{t \to \infty} \left[ \frac{-1}{t} - \frac{-1}{1} \right]
\]

\[
= 4[0 - (-1)]
\]

\[
= 4
\]

Therefore, the original integral in question must converge to a value less than 4.
5. The probability density function (pdf) of the length (in minutes) of phone calls on a wireless network is given by $f(x) = ke^{-0.2x}$ where $x$ is the number of minutes. Note that the domain is $x \geq 0$ since we can’t have a negative number of minutes.

(a) **What must be the value of $k$?**

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

$$
\int_0^\infty ke^{-0.2x} \, dx = \lim_{t \to \infty} \int_0^t ke^{-0.2x} \, dx \\
= \lim_{t \to \infty} \left. \frac{ke^{-0.2x}}{-0.2} \right|_0^t \\
= \lim_{t \to \infty} k e^{-0.2t} - ke^0 \\
= 0 - \frac{k}{-0.2} \\
= 5k
$$

So, we have $5k = 1$ or $k = 0.2$.

(b) **What fraction of calls last more than 3 minutes?**

$$
\int_3^\infty 0.2e^{-0.2x} \, dx = \lim_{t \to \infty} \int_3^t 0.2e^{-0.2x} \, dx \\
= \lim_{t \to \infty} \left. \frac{0.2e^{-0.2x}}{-0.2} \right|_3^t \\
= \lim_{t \to \infty} -e^{-0.2t} - (-e^{-0.6}) \\
= 0 + e^{-0.6} \\
= e^{-0.6} \approx 0.5488
$$

Note that we could instead have computed $1 - \int_0^3 0.2e^{-0.2x} \, dx$ and gotten the same answer.