**Math 106: Review for Exam II**

**INTEGRATION TIPS**

- **Substitution:** usually let $u$ = a function that’s “inside” another function, especially if $du$ (possibly off by a multiplying constant) is also present in the integrand.

- **Parts:** \[ \int u \, dv = uv - \int v \, du \] or \[ \int u'v \, dx = uv - \int u' \, dx \]

  How to choose which part is $u$? Let $u$ be the part that is higher up in the LIATE mnemonic below. (The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter A is replaced by P, which stands for “polynomial.”)

  - Logarithms (such as $\ln x$)
  - Inverse trig (such as $\arctan x$, $\arcsin x$)
  - Algebraic (such as $x$, $x^2$, $x^3 + 4$)
  - Trig (such as $\sin x$, $\cos 2x$)
  - Exponentials (such as $e^x$, $e^{3x}$)

- **Rational Functions (one polynomial divided by another):** if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

- **Partial Fractions:** here’s an illustrative example of the setup.

  \[
  \frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}
  \]

  Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor $(x - 3)$ on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term $(Dx + E)$ here above it on the right.

- **Trigonometric Substitutions:** some suggested substitutions and useful formulae follow.

  - **Radical Form**
    - $\sqrt{a^2 - x^2}$ \( x = a \sin t \)
    - $\sqrt{a^2 + x^2}$ \( x = a \tan t \)
    - $\sqrt{x^2 - a^2}$ \( x = a \sec t \)

  - $\sin^2 x + \cos^2 x = 1$
  - $\tan^2 x + 1 = \sec^2 x$
  - $\cos^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$
  - $\sin(2x) = 2 \sin x \cos x$

- **Powers of Trigonometric Functions:** here are some strategies for dealing with these.

  \[
  \int \sin^m x \cos^n x \, dx \quad \text{Possible Strategy} \quad \text{Identity to Use}
  \]

  \[
  \begin{array}{|c|c|c|}
  \hline
  m \text{ odd} & \text{Break off one factor of } \sin x \text{ and substitute } u = \cos x. & \sin^2 x = 1 - \cos^2 x \\
  \hline
  n \text{ odd} & \text{Break off one factor of } \cos x \text{ and substitute } u = \sin x. & \cos^2 x = 1 - \sin^2 x \\
  \hline
  m \text{ even AND } n \text{ even} & \text{Use } \sin^2 x + \cos^2 x = 1 \text{ to reduce to only powers of } \sin x \text{ or only powers of } \cos x, \text{ then use table of integrals #39–42} & \sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} \\
  & \text{or identities shown to right of this box.} & \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2} \\
  \hline
  \end{array}
  \]
\[
\int \tan^m x \sec^n x \, dx
\]

| \(m\) odd | Break off one factor of \(\sec x \tan x\) and substitute \(u = \sec x\). | \(\tan^2 x = \sec^2 x - 1\) |
| \(n\) even | Break off one factor of \(\sec^2 x\) and substitute \(u = \tan x\). | \(\sec^2 x = \tan^2 x + 1\) |
| \(m\) even AND \(n\) odd | Use identity at right to reduce to powers of \(\sec x\) alone. Then use table of integrals #51 or integration by parts. | \(\tan^2 x = \sec^2 x - 1\) |

**Useful Trigonometric Derivatives and Antiderivatives**

\[
\frac{d}{dx} \tan x = \sec^2 x, \quad \frac{d}{dx} \sec x = \sec x \tan x, \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

- **Improper integrals:** look for \(\infty\) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

  \[
  \lim_{x \to \infty} e^x = \infty \quad \lim_{x \to \infty} e^{-x} = 0 \\
  \lim_{x \to \infty} \frac{1}{x} = 0 \quad \lim_{x \to \infty} \frac{1}{x^2} = 0 \\
  \lim_{x \to \infty} \ln x = \infty \quad \lim_{x \to 0^+} \ln x = -\infty
  \]

1. Evaluate the following.

   (a) \(\int \sin^6 x \cos^3 x \, dx\)

   (b) \(\int \frac{dx}{\sqrt{100 + x^2}}\)
(c) \[ \int_3^\infty \frac{1}{x(\ln x)^{100}} \, dx \]

(d) \[ \int_0^\infty x e^{-2x} \, dx \]

(e) \[ \int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx \]

(f) \[ \int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx \]

(g) \[ \int_{-1}^5 \frac{1}{(x - 1)^6} \, dx \]
2. Find the second-order Taylor polynomial for \( f(x) = \sqrt{x} \) based at \( x_0 = 100 \). Then use your polynomial to estimate \( \sqrt{105} \).

3. What is the largest possible error that could have occurred in your estimate of \( \sqrt{105} \)?

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.
   
   (a) \( \int_{1}^{\infty} \frac{6 + \cos x}{x^{0.99}} \, dx \)
   
   (b) \( \int_{1}^{\infty} \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \)

5. The probability density function (pdf) of the length (in minutes) of phone calls on a wireless network is given by \( f(x) = ke^{-0.2x} \) where \( x \) is the number of minutes. Note that the domain is \( x \geq 0 \) since we can’t have a negative number of minutes.

   (a) What must be the value of \( k \)?

   (b) What fraction of calls last more than 3 minutes?