1. Use integration by parts to find this integral: $\int \sqrt{x} \ln x \, dx$

"L I A R E" suggests try $u = \ln x$, so $v' = \sqrt{x} = x^{1/2}$.
Then $u' = \frac{1}{x}$ and $v = \frac{4}{5} x^{5/4}$

We have $\int \sqrt{x} \ln x \, dx$ becomes

$\int \sqrt{x} \ln x \, dx = \int uv' \, dx$

$= (\ln x) \left( \frac{4}{5} x^{5/4} \right) - \int \frac{4}{5} x^{5/4} \cdot \frac{1}{x} \, dx$

$= (\ln x) \left( \frac{4}{5} x^{5/4} \right) - \frac{4}{5} \int x^{3/4} \, dx$

$= (\ln x) \left( \frac{4}{5} x^{5/4} \right) - \frac{4}{5} \left( \frac{4}{5} x^{5/4} \right) + C$

$= (\ln x) \left( \frac{4}{5} x^{5/4} \right) - \frac{16}{25} x^{5/4} + C$

OR

$\frac{4}{5} x^{5/4} \left( \ln x - \frac{4}{5} \right) + C$
2. Use partial fraction decomposition to find the following integrals:

2A) \( \int \frac{9x - 22}{(x - 3)(x - 2)} \, dx \)

\[
= \int \frac{A}{x - 3} \, dx + \int \frac{B}{x - 2} \, dx \quad \text{where } A \text{ and } B \text{ are found as follows:}
\]

\[
\frac{9x - 22}{(x - 3)(x - 2)} = \frac{A}{x - 3} + \frac{B}{x - 2} \Rightarrow 9x - 22 = A(x - 2) + B(x - 3)
\]

if \( x = 2 \) we get \( 18 - 22 = A \cdot 0 + B(-1) \)

so \( -4 = -B \)

so \( B = 4 \)

if \( x = 3 \) we find \( 27 - 22 = A \cdot 1 + B \cdot 0 \)

so \( 5 = A \)

so \( A = 5 \)

the \( \int \) becomes \( \int \frac{5}{x - 3} \, dx + \int \frac{4}{x - 2} \, dx \)

\[
= \ln|x - 3| + 4 \ln|x - 2| + C
\]

2B) \( \int \frac{3x^4 + 29x^2 + 2x + 40}{x^2 + 8} \, dx \)

we first must carry out the indicated division:

\[
\begin{array}{c|ccccc}
& 3x^2 + 5 \\
\hline
x^2 + 8 & 3x^4 + 29x^2 + 2x + 40 \\
& - (3x^4 + 24x^2) \\
& \hline
& 5x^2 + 2x + 40 \\
& - (5x^2 + 40) \\
& \hline
& 2x \\
\end{array}
\]

so the integral becomes \( \int \frac{3x^2 + 5 + \frac{2x}{x^2 + 8}}{dx} \)

\[
= \frac{x^3 + 5x + \ln|x^2 + 8|}{x^2 + 8} + C
\]

note: if the remainder had had a constant term, the answer would involve a term using \( \arctan(\frac{x}{2}) \) ... lucky for us!
3. Let \( f(x) = x^{4/3} \). (On a calculator, enter such an expression using ( )'s: Use \( X \sim (4/3) \), not \( X \sim 4/3 \))

3A. Find the Taylor polynomial \( P_2(x) \) of degree two in powers of \((x - 8)\) for \( f(x) \). Organize all your work in a table as we've done in class, and keep all numbers involved as fractions (no decimals).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( f^{(k)}(x) )</th>
<th>( f^{(k)}(8) )</th>
<th>( C_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( x^{4/3} )</td>
<td>( 8^{4/3} = 2^4 = 16 )</td>
<td>( 16/0! = 16 )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{4}{3} x^{1/3} )</td>
<td>( \frac{4}{3} \cdot 8^{1/3} = \frac{4}{3} \cdot \frac{2}{3} = \frac{8}{3} )</td>
<td>( (\frac{8}{3})/1! = \frac{8}{3} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{4}{9} x^{-2/3} )</td>
<td>( \frac{4}{9} \cdot 8^{-2/3} = \frac{4}{9} \cdot \frac{1}{8} = \frac{1}{9} )</td>
<td>( (\frac{1}{9})/2! = \frac{1}{18} )</td>
</tr>
<tr>
<td>3</td>
<td>( -\frac{32}{27} x^{-5/3} )</td>
<td>( \text{(used in 3D below for error guarantee)} )</td>
<td></td>
</tr>
</tbody>
</table>

So:

\[
P_2(x) = 16 + \frac{8}{3}(x-8) + \frac{1}{18}(x-8)^2
\]

3B. Use \( P_2(x) \) to approximate \( 11^{4/3} \).

Since \( P_2(x) = f(x) = x^{4/3} \), we set \( x = 11 \) and find

\[
P_2(11) = 16 + \frac{8}{3}(11-8) + \frac{1}{18}(11-8)^2 = 16 + \frac{8}{3} \cdot 3 + \frac{1}{18} \cdot 9
\]

\[
= 16 + 8 + \frac{1}{2} = 24.5
\]

3C. By calculator directly, what is \( 11^{4/3} \) to six places after the decimal point?

\[
24.463781
\]

3D. What is the error guarantee for this result? That is, on the interval \( [8, 11] \), it is guaranteed that \( |f(x) - P_2(x)| \) will be no bigger than what? In your work, find \( K \) to 4 places after the decimal point and show the formula used to find this guarantee.

A rough copy of a plot of \( f^{(3)}(x) = \frac{8}{27} x^{-5/3} \): \( i: 8 \to 11 \), \( -0.00544 \to -0.009239 \)

so on \( [8, 11] \), a value \( k \) satisfying \( K \geq |f^{(3)}(x)| \) \( i: K = 0.0093 \)

and the corresponding error guarantee is \( (0.0093)(11-3)^3 \)

\[
= \frac{0.0093 \times 28}{6} = \frac{0.2511}{6} = 0.04185
\]

(note the difference \( |24.5 - 24.463781| = 0.036219 \), which is indeed less than this).
4. Use the substitution \( x = 3 \sec t \) to find \( \int \frac{dx}{x^2\sqrt{x^2 - 9}} \). Remove all trig functions from your answer to the extent possible.

So with \( x = 3 \sec t \) we have

\[
dx = 3 \sec t \tan t \, dt
\]

AND \( x^2 = 9 \sec^2 t \)

and \( x^2 - 9 = 9 \sec^2 t - 9 = 9(\sec^2 t - 1) = 9 \tan^2 t \)

so with these substitutions the integral becomes

\[
\int \frac{3 \sec t \tan t \, dt}{(9 \sec^2 t)\sqrt{9 \tan^2 t}}
\]

\[
= \int \frac{3 \sec t \tan t \, dt}{(9 \sec^2 t)(3 \tan t)}
\]

\[
= \frac{1}{9} \int \frac{dt}{\sec^2 t}
\]

\[
= \frac{1}{9} \int \frac{dt}{\cos^2 t} = \frac{1}{9} \int \sec t \, dt = \frac{1}{9} \sin t + C
\]

Now, \( x = 3 \sec t \Rightarrow \frac{x}{3} = \sec t = \frac{1}{\cos t} = \frac{\text{hyp}}{\text{adj}} \Rightarrow \)

set up the like this:

\[
\text{hyp} = x \\
\text{adj} = 3
\]

so this side is \( \text{opp} = \sqrt{x^2 - 9} \)

\[
\text{finally}, \quad \frac{1}{9} \sin t + C = \frac{1}{9} \frac{\text{opp}}{\text{hyp}} + C
\]

\[
= \frac{1}{9} \frac{\sqrt{x^2 - 9}}{x} + C
\]
5. Here's a fact: an antiderivative of \( \frac{4}{x^2\sqrt{4 - x^2}} \) is \( -\sqrt{4 - x^2} \); you may find it useful to build this antiderivative into your calculator to answer the following.

5A) The integral \( \int_0^1 \frac{4 \, dx}{x^2\sqrt{4 - x^2}} \) is improper because of the vertical asymptote at \( x = 0 \). Does it converge (and if so, to what value) or diverge? Make a table of values of \( \int_B^{\infty} \frac{4 \, dx}{x^2\sqrt{4 - x^2}} \) using at least 4 well-chosen values of \( B \to 0^+ \) that support your conclusion.

By definition, \( \int_0^1 \frac{4 \, dx}{x^2\sqrt{4 - x^2}} = \lim_{B \to 0^+} \int_B^{1} \frac{4 \, dx}{x^2\sqrt{4 - x^2}} \), provided the limit exists.

\[
\lim_{B \to 0^+} \int_B^{1} \frac{4 \, dx}{x^2\sqrt{4 - x^2}} = \lim_{B \to 0^+} \frac{-\sqrt{4 - x^2}}{x} \bigg|_B^1 = \lim_{B \to 0^+} \left( \frac{-\sqrt{4 - x^2}}{x} \bigg|_B^1 \right)
\]

This table

| 0.5 | 2.14... |
| 0.1 | 18.2429... |
| 0.01 | 199.0465... |
| 0.001 | 1999.0467... |

\( \vdots \)

suggests the limit doesn't exist (or is \( \infty \) which isn't a number)

so the integral \( \int_0^1 \frac{4 \, dx}{x^2\sqrt{4 - x^2}} \) DIVERGES

5B) The integral \( \int_1^2 \frac{4 \, dx}{x^2\sqrt{4 - x^2}} \) is improper because of the vertical asymptote at \( x = 2 \). Does it converge (and if so, to what value) or diverge? Make a table of values of \( \int_1^B \frac{4 \, dx}{x^2\sqrt{4 - x^2}} \) using at least 4 well-chosen values of \( B \to 2^- \) that support your conclusion.

By definition, \( \int_1^2 \frac{4 \, dx}{x^2\sqrt{4 - x^2}} = \lim_{B \to 2^-} \int_1^{B} \frac{4 \, dx}{x^2\sqrt{4 - x^2}} \), provided this limit exists.

Here \( \lim_{B \to 2^-} \int_1^{B} \frac{4 \, dx}{x^2\sqrt{4 - x^2}} = \lim_{B \to 2^-} \frac{-\sqrt{4 - x^2}}{x} \bigg|_1^{B} = \lim_{B \to 2^-} \left( \frac{-\sqrt{4 - x^2}}{x} \bigg|_1^{B} \right) \)

This table:

<table>
<thead>
<tr>
<th>B</th>
<th>( \frac{-\sqrt{4 - x^2}}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1.850133...</td>
</tr>
<tr>
<td>1.9</td>
<td>1.793667...</td>
</tr>
<tr>
<td>1.99</td>
<td>1.731697...</td>
</tr>
<tr>
<td>1.9999</td>
<td>1.7310508...</td>
</tr>
<tr>
<td>1.999999</td>
<td>1.7310508...</td>
</tr>
</tbody>
</table>

\( \sqrt{3} \approx 1.7320508 \)

\( \vdots \)

suggests the limit is \( \sqrt{3} \), so we say

\( \int_1^2 \frac{4 \, dx}{x^2\sqrt{4 - x^2}} = \sqrt{3} \) (or converges to \( \sqrt{3} \))

5C) Does the improper integral \( \int_0^2 \frac{4 \, dx}{x^2\sqrt{4 - x^2}} \) converge or diverge? Explain.

5D) Set your calculator window to \([0, 2.1] \) by \([0, 8] \) and plot \( \frac{4}{x^2\sqrt{4 - x^2}} \); make a good sketch of what you see. Explain why the resulting graph seems to support your answer to 5C.

All the graph suggests is that's (a lot) more area from \([0, 1] \) than from \([1, 2] \).

This indeed agrees with our answer to 5A & 5B (8 hence 5C) since the area from \([0, 1] \)

is unbounded while there's finite area of \( \sqrt{3} \)

Under the graph of \( f(x) = \frac{4}{x^2\sqrt{4 - x^2}} \) on \([1, 2] \)

\( \text{the area is } \sqrt{3} \)
6. Use an appropriate p-test comparison to decide if \( \int_2^\infty \frac{x^3+1}{x^4+4x} \, dx \) converges or diverges. Show all your work.

Since as \( x \to \infty \), the function \( \frac{x^3+1}{x^4+4x} \) looks more and more like \( \frac{1}{x} \),
and \( \int_2^\infty \frac{1}{x} \, dx \) diverges, we might expect \( \int_2^\infty \frac{x^3+1}{x^4+4x} \, dx \) also diverges.

We compare: we need \( \frac{x^3+1}{x^4+4x} \) to be above \( \frac{1}{x} \).

Now, \( \frac{x^3+1}{x^4+4x} \geq \frac{1}{x} \) for \( x \in [2, \infty) \)
\[ \iff x^4 + x \geq x^4 + 4x \]
\[ \iff x \geq 4x \]
\[ \iff 0 \geq 3x. \quad \text{But this is false.} \]

We need to "squish" \( \frac{1}{x} \) by comparing \( \frac{x^3+1}{x^4+4x} \) and \( \frac{\sqrt{x}}{x} \):

Is \( \frac{x^3+1}{x^4+4x} \geq \frac{\sqrt{x}}{x} \)? Yes, go follows:

\[ \iff x^4 + x \geq \frac{1}{2} (x^4 + 4x) \]
\[ \iff x^4 + x \geq \frac{x^4}{2} + 2x \]
\[ \iff \frac{x^4}{2} \geq x \iff x^3 \geq 2 \quad \text{and this is true for } x \in [2, \infty) \]

Now, by our p-test, \( \int_2^\infty \frac{\sqrt{x}}{x} \, dx \) diverges.

And for individual \( x \)'s, \( \frac{x^3+1}{x^4+4x} \geq \frac{\sqrt{x}}{x} \) (for \( x \in [2, \infty) \))

So by the comparison test, \( \int_2^\infty \frac{x^3+1}{x^4+4x} \, dx \) diverges as well.

[It is wrong to write \( \int_2^\infty \frac{x^3+1}{x^4+4x} \, dx \geq \int_2^\infty \frac{\sqrt{x}}{x} \, dx \). Neither expression is a number!]
