NAME:

Instruction: Read each question carefully. Explain ALL your work and give reasons to support your answers.

Advice: DON’T spend too much time on a single problem.

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1. (10 pts.) (a) Find the exact value of the definite integral
\[ \int_{1}^{2} x \ln(x^2) \, dx. \]

Let \( u = \ln(x^2) \) and \( dv = x \, dx \). It follows that \( du = \frac{2}{x^2} \cdot 2x \, dx = \frac{2}{x} \, dx \) and \( v = \frac{x^2}{2} \). Using the techniques of integration by parts, we have
\[
\int_{1}^{2} x \ln(x^2) \, dx = \ln(x^2) \cdot \frac{x^2}{2} \bigg|_{1}^{2} - \int_{1}^{2} \frac{x^2}{2} \cdot \frac{2}{x} \, dx
\]
\[
= [2 \ln 4 - 0] - \frac{x^2}{2} \bigg|_{1}^{2}
\]
\[
= 2 \ln 4 - \frac{3}{2}.
\]

(10 pts.) (b) Evaluate the indefinite integral
\[ \int \frac{2x}{(x+1)(3x-1)} \, dx. \]

Write \( \frac{2x}{(x+1)(3x-1)} = \frac{A}{x+1} + \frac{B}{3x-1} \). Then,
\[
2x \equiv A(3x - 1) + B(x + 1)
\]
\[
\equiv (3A + B)x + (-A + B).
\]
So, \( 2 = 3A + B \) and \( 0 = -A + B \). It follows that \( A = \frac{1}{2} = B \).

Therefore,
\[
\int \frac{2x}{(x+1)(3x-1)} \, dx = \frac{1}{2} \int \frac{dx}{x+1} + \int \frac{dx}{3x-1}
\]
\[
= \frac{1}{2} \ln |x+1| + \frac{1}{6} \ln |3x-1| + C.
\]
2. (10 pts.) Find the indefinite integral
\[ \int \frac{1}{x^2 \sqrt{x^2 - 1}} \, dx. \]

Let \( x = \sec \theta \) so that \( dx = \sec \theta \tan \theta d\theta \). Now, \( \sqrt{x^2 - 1} = \tan \theta \) and \( x^2 = \sec^2 \theta \) so
\[
\int \frac{1}{x^2 \sqrt{x^2 - 1}} \, dx = \int \frac{\sec \theta \tan \theta d\theta}{\sec^2 \theta \tan \theta} = \int \cos \theta \, d\theta = \sin \theta + C = \frac{\sqrt{x^2 - 1}}{x} + C.
\]

(10 pts.) (b) Find the indefinite integral
\[ \int \frac{1}{x^2 - 2x + 5} \, dx. \]

Note that (completing the squares) \( x^2 - 2x + 5 = (x^2 - 2x + 1) + 4 = (x - 1)^2 + 2^2 \). Now, let \( w = \frac{(x-1)}{2} \) so \( dw = \frac{dx}{2} \) or \( dx = 2dw \). Thus,
\[
\int \frac{1}{x^2 - 2x + 5} \, dx = \int \frac{2dw}{2^2(w^2 + 1)} = \frac{1}{2} \int \frac{dw}{w^2 + 1}
= \frac{1}{2} \arctan w + C = \frac{1}{2} \arctan \left( \frac{x - 1}{2} \right) + C.
\]
3. Suppose a function $f$ satisfies

$$f(1) = 1, f'(1) = -3, f''(1) = 2, f'''(1) = 3.$$ 

(10 pts.) (a) Write down the third-degree Taylor polynomial $P_3(x)$ for $f$ based at $x_0 = 1$.

The third degree Taylor polynomial is given by

$$P_3(x) = f(1) + f'(1)(x - 1) + f''(1)\frac{(x - 1)^2}{2!} + f'''(1)\frac{(x - 1)^3}{3!}$$

$$= 1 - 3(x - 1) + (x - 1)^2 + \frac{(x - 1)^3}{2}.$$

(5pts.) (b) Suppose it is known that for $0 \leq x \leq 2$, $|f^{(4)}(x)| \leq 0.5$. What is the maximum possible error committed by using $P_3(x)$ to estimate $f(x)$ for $0 \leq x \leq 2$?

The maximum error is given by $\frac{K_4}{4!}|x - 1|^4$ where $K_4$ is the bound 0.5. For $0 \leq x \leq 2$, $|x - 1|^4 \leq 1$ so that the maximum error is $\frac{0.5}{4!} = \frac{1}{48}$.

(5 pts.) (c) Suppose $g(x) = f(x + 1)$. Use part (a) to find the third-degree Maclaurin polynomial for $g$.

Since $g(x) = f(x + 1)$, it follows that $g^{(k)}(x) = f^{(k)}(x + 1)$. In particular, we have $g^{(k)}(0) = f^{(k)}(1)$. Thus,

$$M_3(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3$$

$$= 1 - 3x + x^2 + \frac{x^3}{2}.$$
4. (10 pts.) (a) Let \( f(x) = \cos(2x) \). Find the fourth-degree Maclaurin polynomial \( M_4(x) \) for \( f \).

First, we have \( f'(x) = -2\sin(2x) \), \( f''(x) = -4\cos(2x) \), \( f'''(x) = 8\sin(2x) \) and \( f^{(4)}(x) = 16\cos(2x) \). It follows that \( f(0) = 1 \), \( f'(0) = 0 \), \( f''(0) = -4 \), \( f'''(0) = 0 \), \( f^{(4)}(0) = 16 \). Thus,

\[
M_4(x) = 1 - \frac{4}{2!} x^2 + \frac{16}{4!} x^4 = 1 - 2x^2 + \frac{2}{3} x^4.
\]

(10 pts.) (b) Solve the following Initial Value Problem

\[
y' = (1 + x^2)e^y, \quad y(0) = 0.
\]

Separating the variables yields

\[
\frac{dy}{e^y} = (1 + x^2)dx
\]

or

\[
\int e^{-y} \ dy = \int (1 + x^2) \ dx.
\]

It follows that \(-e^{-y} = x + \frac{x^3}{3} + C\). The initial condition \( y(0) = 0 \) implies that \( C = -1 \) so that \( e^{-y} = 1 - x - \frac{x^3}{3} \). It follows that

\[
y = -\ln|1 - x - \frac{x^3}{3}|.
\]
5. Determine whether each of the following improper integrals converges or diverges. Justify your answers.

(10 pts.) (a) \[ \int_1^\infty \frac{1}{(1+x^3)^2} \, dx \]

[Hint: compare this integral with another improper integral]

For \( x \geq 1, (1 + x^3) > (x^3)^2 = x^6 \). Thus, for any number \( b \geq 1 \), we have

\[ \int_1^b \frac{1}{(1 + x^3)^2} \, dx < \int_1^b \frac{1}{x^6} \, dx = \frac{1}{5} x^{-5} \bigg|_1^b = \frac{1}{5} - b^{-5}. \]

Passing to the limit as \( b \to \infty \), we have

\[ \int_1^\infty \frac{1}{(1+x^3)^2} \, dx = \lim_{b \to \infty} \int_1^b \frac{1}{(1+x^3)^2} \, dx < \lim_{b \to \infty} \frac{1}{5} - b^{-5} = \frac{1}{5}. \]

Thus, \( \int_1^\infty \frac{1}{(1+x^3)^2} \, dx \) converges.

(10 pts.) (b) \[ \int_0^1 \frac{1}{2x-1} \, dx \]

This integral is improper since the integrand is not defined at \( x = 1/2 \). We write

\[ \int_0^1 \frac{1}{2x-1} \, dx = \int_0^{1/2} \frac{1}{2x-1} \, dx + \int_{1/2}^1 \frac{1}{2x-1} \, dx. \]

Now, the improper integral

\[ \int_0^{1/2} \frac{1}{2x-1} \, dx = \lim_{b \to 1/2^-} \int_0^b \frac{1}{2x-1} \, dx \]

\[ = \lim_{b \to 1/2^-} \frac{1}{2} \ln|2x-1| \bigg|_0^b \]

\[ = \lim_{b \to 1/2^-} \frac{1}{2} \ln|2b-1| \text{ which does not exist.} \]

Since one of the two improper integrals does not exist the original improper cannot exist and so it diverges.