Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

*Advice:* DON’T spend too much time on a single problem.

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1. Evaluate each of the following indefinite integrals (be sure to indicate what techniques you use).

(10 pts.) (a)  
\[ \int x^2 \cos(2x^3) \, dx. \]
Let \( u = 2x^3 \). Then \( du = 6x^2 \, dx \) or \( x^2 \, dx = \frac{1}{6} \, du \). It follows that
\[ \int x^2 \cos(2x^3) \, dx = \int \cos(2x^3) \cdot x^2 \, dx = \int \cos u \cdot \frac{1}{6} \, du = \frac{1}{6} \sin u + C = \frac{\sin(2x^3)}{6} + C. \]

(10 pts.) (b)  
\[ \int \ln x \, dx. \]
Let \( u = \ln x \) and \( dv = x^{-1/2} \, dx \). Then, \( v = 2x^{1/2} \) and \( du = \frac{dx}{x} \). It follows that
\[ \int \frac{\ln x \, dx}{\sqrt{x}} \overset{\text{IBP}}{=} \ln x \cdot 2x^{1/2} - \int 2x^{1/2} \cdot \frac{1}{x} \, dx = 2\sqrt{x} \ln x - 2 \int x^{-1/2} \, dx = 2\sqrt{x} \ln x - 2 \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} \ln x - 4\sqrt{x} + C = 2\sqrt{x}(\ln x - 2) + C. \]
2. Evaluate each of the following indefinite integrals (be sure to indicate what techniques you use).

(10 pts.)(a) 
\[ \int \frac{dt}{t^2 \sqrt{t^2 - 1}}. \]

Let \( t = \sec \theta \) so that \( t^2 = \sec^2 \theta, \sqrt{t^2 - 1} = \tan \theta \) and \( dt = \sec \theta \tan \theta \, d\theta \). It follows that
\[ \int \frac{dt}{t^2 \sqrt{t^2 - 1}} = \int \frac{\sec \theta \tan \theta \, d\theta}{\sec^2 \theta \tan \theta} = \int \cos \theta \, d\theta = \sin \theta + C = \frac{\sqrt{t^2 - 1}}{t} + C. \]

That \( \sin \theta = \frac{\sqrt{t^2 - 1}}{t} \) follows from the right angled triangle associated to the substitution \( t = \sec \theta \).

(10 pts.)(b) 
\[ \int \frac{x^3 + 2x^2 - 3x + 4}{x^2 + 2x - 3} \, dx. \]

First, using long division, we have \( \frac{x^3 + 2x^2 - 3x + 4}{x^2 + 2x - 3} = x + \frac{4}{x^2 + 2x - 3} \). Now, we write
\[ \frac{4}{x^2 + 2x - 3} = \frac{4}{(x + 3)(x - 1)} = \frac{A}{x + 3} + \frac{B}{x - 1}. \]
Thus, \( 4 \equiv A(x - 1) + B(x + 3) \).

At \( x = 1 \), we have \( 4 = 4B \Rightarrow B = 1 \). At \( x = -3 \), we have \( 4 = -4A \Rightarrow A = -1 \). It follows that
\[ \int \frac{x^3 + 2x^2 - 3x + 4}{x^2 + 2x - 3} \, dx = \int x - \frac{1}{x + 3} + \frac{1}{x - 1} \, dx = \frac{x^2}{2} - \ln |x + 3| + \ln |x - 1| + C. \]
3. Evaluate each of the following improper integrals.

(10 pts.) (a) \[ \int_{-\infty}^{1} \frac{dx}{(2x - 3)^3} \]

First,

\[ \int_{-\infty}^{1} \frac{dx}{(2x - 3)^3} = \lim_{b \to -\infty} \int_{b}^{1} \frac{dx}{(2x - 3)^3}. \]

Let \( u = 2x - 3 \) so that \( du = 2 \, dx \) or \( dx = \frac{du}{2} \). Now

\[ \int \frac{dx}{(2x - 3)^3} = \frac{1}{2} \int u^{-3} \, du \]

\[ = \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C = -\frac{1}{4} u^{-2} + C \]

\[ = -\frac{1}{4} (2x - 3)^{-2} + C. \]

This implies that

\[ \int_{-\infty}^{1} \frac{dx}{(2x - 3)^3} = \lim_{b \to -\infty} \left[ -\frac{1}{4} (2x - 3)^{-2} \right]_{b}^{1} = \lim_{b \to -\infty} \left[ -\frac{1}{4} + \frac{1}{4} (2b - 3)^{-2} \right] = -\frac{1}{4}. \]

(10 pts.) (b) \[ \int_{2}^{3} \frac{x}{\sqrt{3 - x}} \, dx \]

Let \( u = 3 - x \). Thus, \( du = -dx \) and \( x = 3 - u \). It follows that

\[ \int_{2}^{3} \frac{x}{\sqrt{3 - x}} \, dx = \int_{-1}^{0} \frac{3 - u}{\sqrt{u}} \, du = \int_{0}^{1} \frac{3 - u}{\sqrt{u}} \, du \]

\[ = \lim_{b \to 0} \int_{b}^{1} 3u^{-1/2} - u^{1/2} \]

\[ = \lim_{b \to 0} \left[ \frac{3u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2} \right]_{b}^{1} \]

\[ = \lim_{b \to 0} \left( 6 \cdot \frac{2}{3} - \left( \frac{3b^{1/2}}{1/2} - \frac{b^{3/2}}{3/2} \right) \right) \]

\[ = 6 \cdot \frac{2}{3} = \frac{16}{3}. \]
4. Let \( f(x) = e^{2x} \).

(8 pts.) (a) Find the third-order Taylor polynomial \( P_3(x) \) of \( f(x) \) based at \( x_0 = 1 \).

**Note that** \( f(x) = e^{2x}, f'(x) = 2e^{2x}, f''(x) = 4e^{2x} \) and \( f'''(x) = 8e^{2x} \). It follows that \( f(1) = e^2, f'(1) = 2e^2, f''(1) = 4e^2 \) and \( f'''(1) = 8e^2 \). Thus the desired Taylor polynomial is given by

\[
P_3(x) = e^2 + 2e^2(x - 1) + \frac{4e^2}{2}(x - 1)^2 + \frac{8e^2}{3!}(x - 1)^3
\]

\[
= e^2[1 + 2(x - 1) + 2(x - 1)^2 + \frac{4}{3}(x - 1)^3].
\]

(8 pts.) (b) Find the third-order Maclaurin polynomial \( M_3(x) \) of \( f(x) \).

Using the calculation of the first three derivatives of \( f \) from part (a), we have \( f(0) = 1, f'(0) = 2, f''(0) = 4, f'''(0) = 8 \). It follows that

\[
M_3(x) = 1 + 2x + \frac{4}{2}x^2 + \frac{8}{3!}x^3 = 1 + 2x + 2x^2 + \frac{4}{3}x^3.
\]

(4 pts.) (c) What is the maximum error committed by using \( M_3(x) \) (as in part (b)) over the interval \([-1, 1]\), according to Taylor’s Theorem? [Hint: how do you obtain \( K_4 \)?]

By Taylor’s theorem, we know that

\[
|M_3(x) - f(x)| \leq \frac{K_4}{4!}|x - 0|^4
\]

where \( K_4 \) is a constant such that \( |f^{(4)}(x)| \leq K_4 \) for all \( x, -1 \leq x \leq 1 \). Note that \( f'''(x) = 8e^{2x} \) thus \( f^{(4)}(x) = 16e^{2x} \). Therefore, over the interval \([-1, 1]\), \( |f^{(4)}(x)| \leq 16e^2 \) so that we can choose \( K_4 = 16e^2 \). We can conclude that the maximum error committed by \( M_3 \), using Taylor’s theorem, is

\[
\frac{K_4}{4!} \cdot 1 = \frac{16e^2}{24} = \frac{2e^2}{3}.
\]
5. (12 pts.) (a) Use comparison to determine whether the following improper integral converges or diverges. Justify your answer.

\[ \int_1^\infty \frac{e^{-x}}{\sqrt{x^3+1}} \, dx \]

For \( x \geq 1, \sqrt{x^3+1} > 1 \) so that \( \frac{1}{\sqrt{x^3+1}} < 1 \). It follows that

\[ \frac{e^{-x}}{\sqrt{x^3+1}} < e^{-x} \quad \text{and} \quad \int_1^\infty \frac{e^{-x}}{\sqrt{x^3+1}} \, dx < \int_1^\infty e^{-x} \, dx, \text{which converges.} \]

This comparison implies that the improper integral \( \int_1^\infty \frac{e^{-x}}{\sqrt{x^3+1}} \, dx \) converges.

(b) Consider the following function

\[ f(x) = \begin{cases} \frac{k}{12}x^2(12-x), & \text{for } 0 \leq x \leq 12; \\ 0, & \text{otherwise.} \end{cases} \]

For what value of \( k \) is \( f(x) \) a probability density function?

For \( f(x) \) to be a p.d.f., we must have (1) \( f(x) \geq 0 \) and (2) \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \). Note that

\[ \int_{-\infty}^{\infty} f(x) \, dx = \int_0^{12} kx^2(12-x) \, dx = k \int_0^{12} 12x^2 - x^3 \, dx = k \left[ 4x^3 - \frac{x^4}{4} \right]_0^{12} = k \cdot \left( 4 \cdot (12)^3 - \frac{(12)^4}{4} \right) = (12)^3 k. \]

Thus, for \( f(x) \) to be a p.d.f., \( (12)^3 k = 1 \) or \( k = \frac{1}{(12)^3} \).