(a) Find the third-order Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x}$ centered at $x_0 = 4$.

Since $f(x) = \sqrt{x} = x^{1/2}$, it follows that $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$. Thus, at $x_0 = 4$, we have $f(4) = 2$, $f'(4) = \frac{1}{4}$, $f''(4) = -\frac{1}{32}$, $f'''(4) = \frac{3}{256}$. Now, the third Taylor polynomial is given by

$$P_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3$$

$$= 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2 + \frac{1}{512}(x - 4)^3.$$ 

(b) Consider the function $g(x) = e^{2x}$. Using Taylor’s theorem, estimate the error committed by the 4th degree Maclaurin polynomial $M_4(x)$ for $g(x)$ over the interval $[-1, 1]$.

Since $g(x) = e^{2x}$, it follows that $g'(x) = 2e^{2x}$, $g''(x) = 4e^{2x}$, $g'''(x) = 8e^{2x}$, $g^{(4)}(x) = 16e^{2x}$, and $g^{(5)}(x) = 32e^{2x}$. Over the interval $[-1, 1]$, the function $e^{2x}$ is increasing so that the maximum occurs at the right endpoint $x = 1$. Thus, we can choose $K_5 = 32e^2$. Again, $|x - 0| < 1$ for all $x$, $-1 \leq x \leq 1$. It follows from Taylor’s theorem that

$$|g(x) - M_4(x)| \leq \frac{K_5}{5!}|x - 0|^5$$

$$\leq \frac{32e^2}{5!} \cdot 1^5 \approx 1.9704.$$ 

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