INTEGRATION TIPS

- **Substitution:** usually let \( u \) = a function that’s “inside” another function, especially if \( du \) (possibly off by a multiplying constant) is also present in the integrand.

- **Parts:**
  \[
  \int u \, dv = uv - \int v \, du \\
  \text{or} \quad \int uv' \, dx = uv - \int u'v \, dx
  \]
  
  How to choose which part is \( u \)? Let \( u \) be the part that is higher up in the LIATE mnemonic below.

  *(The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter A is replaced by P, which stands for “polynomial.”)*

- **Logarithms (such as ln \( x \))**
- **Inverse trig (such as arctan \( x \), arcsin \( x \))**
- **Algebraic (such as \( x, x^2, x^3 + 4 \))**
- **Trig (such as \( \sin x, \cos 2x \))**
- **Exponentials (such as \( e^x, e^{3x} \))**

- **Rational Functions (one polynomial divided by another):** if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

- **Partial Fractions:** here’s an illustrative example of the setup.
  \[
  \frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}
  \]
  
  Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor \((x - 3)\) on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term \((Dx + E)\) above it on the right.

- **Trigonometric Substitutions:** some suggested substitutions and useful formulae follow.

<table>
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<tr>
<th>Radical Form</th>
<th>( \sqrt{a^2 - x^2} )</th>
<th>( \sqrt{a^2 + x^2} )</th>
<th>( \sqrt{x^2 - a^2} )</th>
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<tr>
<td>Substitution</td>
<td>( x = a \sin t )</td>
<td>( x = a \tan t )</td>
<td>( x = a \sec t )</td>
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- **Trigonometric Functions:** here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>( \int \sin^m x \cos^n x , dx )</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
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<tr>
<td>( m ) odd</td>
<td>Break off one factor of ( \sin x ) and substitute ( u = \cos x ).</td>
<td>( \sin^2 x = 1 - \cos^2 x )</td>
</tr>
<tr>
<td>( n ) odd</td>
<td>Break off one factor of ( \cos x ) and substitute ( u = \sin x ).</td>
<td>( \cos^2 x = 1 - \sin^2 x )</td>
</tr>
<tr>
<td>( m ) even AND ( n ) even</td>
<td>Use ( \sin^2 x + \cos^2 x = 1 ) to reduce to only powers of ( \sin x ) or only powers of ( \cos x ), then use table of integrals #39–42.</td>
<td>( \sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} ) or identities shown to right of this box.</td>
</tr>
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</table>
\[ \int \tan^m x \sec^n x \, dx \quad \text{Possible Strategy} \quad \text{Identity to Use} \]

| \( m \) odd | Break off one factor of \( \sec x \tan x \) and substitute \( u = \sec x \). | \( \tan^2 x = \sec^2 x - 1 \) |
| \( n \) even | Break off one factor of \( \sec^2 x \) and substitute \( u = \tan x \). | \( \sec^2 x = \tan^2 x + 1 \) |
| \( m \) even AND \( n \) odd | Use identity at right to reduce to powers of \( \sec x \) alone. Then use table of integrals #51 or integration by parts. | \( \tan^2 x = \sec^2 x - 1 \) |

Useful Trigonometric Derivatives and Antiderivatives

\[
\frac{d}{dx} \tan x = \sec^2 x \quad \frac{d}{dx} \sec x = \sec x \tan x \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

- Improper integrals: look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

\[
\lim_{x \to \infty} e^x = \infty \\
\lim_{x \to \infty} e^{-x} = 0 \\
\lim_{x \to \infty} \frac{1}{x} = 0 \\
\lim_{x \to \infty} \frac{1}{x^2} = 0 \\
\lim_{x \to 0^+} \ln x = \infty \\
\lim_{x \to 0^-} \ln x = -\infty \\
\lim_{x \to \infty} \arctan x = \pi/2
\]

1. Evaluate the following.

(a) Let \( u = \sin x \), so \( du = \cos x \, dx \).

\[
\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx \\
= \int u^6 (1 - u^2) \, du \\
= \int (u^6 - u^8) \, du \\
= \frac{u^7}{7} - \frac{u^9}{9} + C \\
= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C
\]

Use \( \cos^2 x = 1 - \sin^2 x \).

(b) Let \( x = 10 \tan t \), so \( dx = 10 \sec^2 t \, dt \).

[Diagram]

\[
x^2 + 10^2 = y^2 \Rightarrow y = \sqrt{x^2 + 100} \\
\sec t = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2 + 100}}{10} \\
\tan t = \frac{\text{opp}}{\text{adj}} = \frac{x}{10}
\]
\[ \int \frac{dx}{\sqrt{100 + x^2}} = \int \frac{10 \sec^2 t \, dt}{\sqrt{100 + 100 \tan^2 t}} \]
\[ = \int \frac{10 \sec^2 t \, dt}{10 \sqrt{1 + \tan^2 t}} \]
\[ = \int \frac{\sec^2 t \, dt}{\sqrt{1 + \tan^2 t}} \]
\[ = \ln |\sec t + \tan t| + C \]
\[ = \ln \left| \sqrt{x^2 + 100} + \frac{x}{10} \right| + C \]

Now use \(1 + \tan^2 t = \sec^2 t\).

\[ (c) \] This is an improper integral, so we need to use a limit.

\[ \int_3^\infty \frac{1}{x(\ln x)^{100}} \, dx = \lim_{t \to \infty} \int_3^t \frac{1}{x(\ln x)^{100}} \, dx \]
\[ = \lim_{t \to \infty} \int_3^t \frac{1}{u^{100}} \, du \]
\[ = \lim_{t \to \infty} \left[ \frac{-1}{99(\ln u)^{99}} \right]_3^t \]
\[ = \lim_{t \to \infty} \left[ \frac{-1}{99(\ln t)^{99}} - \frac{-1}{99(\ln 3)^{99}} \right] \]
\[ = 0 - \frac{-1}{99(\ln 3)^{99}} \]
\[ = \frac{1}{99(\ln 3)^{99}} \]

So, the integral converges (to this value).

\[ (d) \] We’ll use integration by parts: \( u = x \Rightarrow du = dx \) and \( dv = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2} \).

\[ \int_0^\infty xe^{-2x} \, dx = \lim_{t \to \infty} \int_0^t xe^{-2x} \, dx \]
\[ = \lim_{t \to \infty} \left[ xe^{-2x} \right]_0^t - \frac{1}{2} \int_0^t e^{-2x} \, dx \]
\[ = \lim_{t \to \infty} \left[ xe^{-2x} \right]_0^t - \frac{1}{4} \int_0^t e^{-2x} \, dx \]
\[ = \lim_{t \to \infty} \left[ \frac{-x}{2e^{2t}} - \frac{1}{4e^{2t}} \right]_0^t \]
\[ = \lim_{t \to \infty} \left[ \frac{-t}{2e^{2t}} - \frac{1}{4e^{2t}} \right] - \left[ \frac{0}{2e^0} - \frac{1}{4e^0} \right] \]
\[ = (0 - 0) - (0 - 1/4) \]
\[ = 1/4 \]

So, the integral converges (to this value).

\[ (e) \] Partial Fractions:

Write \( \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 3} \). Now multiply both sides by \((x - 3)(x^2 + 1)\) to get
\[3x^2 + 2x - 13 = (Ax + B)(x - 3) + C(x^2 + 1).\]

Let \(x = 3\). Then \(20 = C(10)\), so \(C = 2\).  
Let \(x = 0\). Then \(-13 = B(-3) + 2(1)\), so \(B = 5\).  
Let \(x = 1\). Then \(-8 = (A(1) + 5)(-2) + 2(2)\), so \(A = 1\).

\[
\int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx = \int \left[ \frac{x + 5}{x^2 + 1} + \frac{2}{x - 3} \right] \, dx
\]

Let \(u = x^2 + 1\), so \(du = 2x \, dx\).

\[
\int \left\{ \frac{1}{u} + \int \left[ \frac{5}{x^2 + 1} + \frac{2}{x - 3} \right] \, dx \right\} \, du = \ln u + 5 \arctan x + 2 \ln |x - 3| + D
\]

\[
= \ln(x^2 + 1) + 5 \arctan x + 2 \ln |x - 3| + D
\]

(f) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

\[
\frac{4x^2 - 3x + 2}{x - 6} = \frac{4x^3 - 27x^2 + 20x - 17}{4x^3 - 24x^2 - 3x^2 + 18x - 2x - 12} = \frac{-5}{-5}
\]

Now, we compute the integral.

\[
\int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx = \int \left[ 4x^2 - 3x + 2 - \frac{5}{x - 6} \right] \, dx = \frac{4x^3}{3} - \frac{3x^2}{2} + 2x - 5 \ln |x - 6| + C
\]

(g) This integral is improper at \(x = 1\) because the integrand has a vertical asymptote there, so we split into two integrals.

\[
\int_{-1}^{0} \frac{1}{(x - 1)^6} \, dx = \int_{-1}^{1} \frac{dx}{(x - 1)^6} + \int_{1}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to 1^-} \int_{-1}^{a} \frac{dx}{(x - 1)^6} + \lim_{b \to 1^+} \int_{b}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to 1^-} \frac{-1}{5(x - 1)^5} \bigg|_{-1}^{a} + \lim_{b \to 1^+} \frac{-1}{5(x - 1)^5} \bigg|_{b}^{5}
\]

\[
= \lim_{a \to 1^-} \left[ -\frac{1}{5(a - 1)^5} - \frac{-1}{5(1 - 1)^5} \right] + \lim_{b \to 1^+} \left[ -\frac{1}{5(5 - 1)^5} - \frac{-1}{5(b - 1)^5} \right]
\]

Since \(\lim_{a \to 1^-} \frac{-1}{5(a - 1)^5} = \infty\) and \(\lim_{b \to 1^+} \frac{-1}{5(b - 1)^5} = \infty\), this integral diverges (to \(\infty\)).
2. Find the second-order Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x = 100$. Then use your polynomial to estimate $\sqrt{105}$.

$$
f(x) = x^{1/2} \quad \quad \quad \quad f(100) = 10
$$

$$
f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \quad \quad \quad f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}
$$

$$
f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x^{3/2}} \quad \quad \quad \quad f''(100) = -\frac{1}{4\cdot100^{3/2}} = -\frac{1}{4000}
$$

$$
P_2(x) = f(100) + f'(100)(x - 100) + \frac{f''(100)}{2!}(x - 100)^2
$$

$$
= 10 + \frac{x - 100}{20} - \frac{(x - 100)^2}{8000}
$$

Now, $\sqrt{105} \approx P_2(105) = 10 + \frac{105 - 100}{20} - \frac{(105 - 100)^2}{8000} = \frac{3279}{320}$

3. What is the largest possible error that could have occurred in your estimate of $\sqrt{105}$?

We know that $|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n + 1)!}|x - x_0|^{n+1}$.

In this case, $n = 2$, $x_0 = 100$, and $x = 105$.

$$
K_3 = \text{max of } |f'''(x)| \text{ on } [100, 105] = \text{max of } \left|\frac{3}{8x^{5/2}}\right| \text{ on } [100, 105] = \frac{3}{8 \cdot 100^{5/2}} = \frac{3}{800,000}
$$

Putting this all together, we have $|f(x) - P_2(x)| \leq \frac{3}{800,000} |105 - 100|^3 = \frac{1}{12800}$

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) $\int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx$

For all $x \geq 1$, we have $\frac{6 + \cos x}{x^{0.99}} \geq \frac{6 - 1}{x^{0.99}} = \frac{5}{x^{0.99}}$ because the minimum value of $\cos x$ is $-1$.

Since $\int_1^\infty \frac{5 \, dx}{x^{0.99}}$ diverges (compute yourself or notice that $p = 0.99 < 1$), we know that the integral in question must diverge too.

(b) $\int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx$

For all $x \geq 1$, we have $\frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^5} = \frac{4}{x^2}$. (We’ve made the denominator smaller and the numerator larger, so the new fraction is larger.)

$$
4 \int_1^\infty \frac{dx}{x^2} = 4 \lim_{t \to \infty} \int_1^t \frac{dx}{x^2}
$$

$$
= 4 \lim_{t \to \infty} \left[ -\frac{1}{x} \right]_1^t
$$

$$
= 4 \lim_{t \to \infty} \left[ -\frac{1}{t} - \frac{1}{1} \right]
$$

$$
= 4[0 - (-1)]
$$

Therefore, the original integral in question must converge to a value less than 4.
(c) \[ \int_{0}^{1} \frac{9}{\sqrt{x^3 + x}} \, dx \]

For \( 0 < x \leq 1 \), we have \( \frac{9}{\sqrt{x^3 + x}} \leq \frac{9}{\sqrt{x}} = 9 \frac{1}{x^{1/2}} \). (We’ve made the denominator smaller, so the new fraction is larger.)

\[
9 \int_{0}^{1} \frac{1}{x^{1/2}} \, dx = 9 \lim_{t \to 0^+} \left[ x^{1/2} \right]_{t}^{1} = 9 \lim_{t \to 0^+} \left[ \frac{1}{2} - \frac{t^{1/2}}{1/2} \right] = 9(2 - 0) = 18
\]

Therefore, the original integral in question must converge to a value less than 18.

5. The probability density function (pdf) of the length (in minutes) of phone calls on a wireless network is given by \( f(x) = ke^{-0.2x} \) where \( x \) is the number of minutes. Note that the domain is \( x \geq 0 \) since we can’t have a negative number of minutes.

(a) What must be the value of \( k \)?

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

\[
\int_{0}^{\infty} ke^{-0.2x} \, dx = \lim_{t \to \infty} \int_{0}^{t} ke^{-0.2x} \, dx = \lim_{t \to \infty} \left. ke^{-0.2x} \right|_{0}^{t} = \lim_{t \to \infty} \left( ke^{-0.2t} - k \right) = 0 - k \left( -0.2 \right) = 5k
\]

So, we have \( 5k = 1 \) or \( k = 0.2 \).

(b) What fraction of calls last more than 3 minutes?

\[
\int_{3}^{\infty} 0.2e^{-0.2x} \, dx = \lim_{t \to \infty} \int_{3}^{t} 0.2e^{-0.2x} \, dx = \lim_{t \to \infty} \left. 0.2e^{-0.2x} \right|_{3}^{t} = \lim_{t \to \infty} \left( -e^{-0.2t} - \left( -e^{-0.6} \right) \right) = 0 + e^{-0.6} = e^{-0.6} \approx 0.5488
\]

Note that we could instead have computed \( 1 - \int_{0}^{3} 0.2e^{-0.2x} \, dx \) and gotten the same answer.