INTRODUCTION TIPS

- **Substitution:** usually let $u = a$ function that's “inside” another function, especially if $du$ (possibly off by a multiplying constant) is also present in the integrand.

- **Parts:**

  $$\int u \, dv = uv - \int v \, du$$

  or

  $$\int uv' \, dx = uv - \int u'v \, dx$$

  How to choose which part is $u$? Let $u$ be the part that is higher up in the LIATE mnemonic below. (The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter A is replaced by P, which stands for “polynomial.”)

  **Logarithms** (such as $\ln x$)
  **Inverse trig** (such as $\arctan x, \arcsin x$)
  **Algebraic** (such as $x, x^2, x^3 + 4$)
  **Trig** (such as $\sin x, \cos 2x$)
  **Exponentials** (such as $e^x, e^{3x}$)

- **Rational Functions** (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

  **Partial Fractions:** here’s an illustrative example of the setup.

  $$\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}$$

  Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor $(x - 3)^2$ on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term $(Dx + E)$ above it on the right.

- **Trigonometric Substitutions:** some suggested substitutions and useful formulae follow.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>$\sqrt{a^2 - x^2}$</th>
<th>$\sqrt{a^2 + x^2}$</th>
<th>$\sqrt{x^2 - a^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substitution</td>
<td>$x = a \sin t$</td>
<td>$x = a \tan t$</td>
<td>$x = a \sec t$</td>
</tr>
</tbody>
</table>

  $$\sin^2 x + \cos^2 x = 1$$
  $$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$$
  $$\tan^2 x + 1 = \sec^2 x$$
  $$\cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$$
  $$\sin(2x) = 2 \sin x \cos x$$

- **Powers of Trigonometric Functions:** here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>$\int \sin^m x \cos^n x , dx$</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ odd</td>
<td>Break off one factor of $\sin x$ and substitute $u = \cos x$.</td>
<td>$\sin^2 x = 1 - \cos^2 x$</td>
</tr>
<tr>
<td>$n$ odd</td>
<td>Break off one factor of $\cos x$ and substitute $u = \sin x$.</td>
<td>$\cos^2 x = 1 - \sin^2 x$</td>
</tr>
<tr>
<td>$m$ even AND $n$ even</td>
<td>Use $\sin^2 x + \cos^2 x = 1$ to reduce to only powers of $\sin x$ or only powers of $\cos x$, then use table of integrals #39–42 or identities shown to right of this box.</td>
<td>$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$</td>
</tr>
</tbody>
</table>
\[
\int \tan^m x \sec^n x \, dx
\]

| \( m \text{ odd} \) | Break off one factor of \( \sec x \tan x \) and substitute \( u = \sec x \). | \( \tan^2 x = \sec^2 x - 1 \) |
| \( n \text{ even} \) | Break off one factor of \( \sec^2 x \) and substitute \( u = \tan x \). | \( \sec^2 x = \tan^2 x + 1 \) |
| \( m \text{ even AND } n \text{ odd} \) | Use identity at right to reduce to powers of \( \sec x \) alone. Then use table of integrals #51 or integration by parts. | \( \tan^2 x = \sec^2 x - 1 \) |

Useful Trigonometric Derivatives and Antiderivatives

\[
\frac{d}{dx} \tan x = \sec^2 x \quad \frac{d}{dx} \sec x = \sec x \tan x \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

- Improper integrals: look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

\[
\lim_{x \to \infty} e^x = \\
\lim_{x \to \infty} e^{-x} = \quad \text{Note: this is the same as } \lim_{x \to -\infty} e^x \\
\lim_{x \to \infty} 1/x = \quad \text{Note: the answer is the same for } \lim_{x \to \infty} 1/x^2 \text{ and similar functions} \\
\lim_{x \to 0^+} 1/x = \quad \text{Note: the answer is the same for } \lim_{x \to 0^+} 1/x^2 \text{ and similar functions} \\
\lim_{x \to \infty} \ln x = \\
\lim_{x \to 0^+} \ln x = \\
\lim_{x \to -\infty} \arctan x = \\
\lim_{x \to \infty} \arctan x =
\]

1. Evaluate the following.

(a) \( \int \sin^6 x \cos^3 x \, dx \)

(b) \( \int \frac{dx}{\sqrt{100 + x^2}} \)
(c) \[ \int_3^\infty \frac{1}{x(\ln x)^{100}} \, dx \]

(d) \[ \int_0^\infty x e^{-2x} \, dx \]

(e) \[ \int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx \]

(f) \[ \int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx \]

(g) \[ \int_{-1}^5 \frac{1}{(x - 1)^6} \, dx \]
2. Find the second-order Taylor polynomial for $f(x) = \sqrt{x}$ centered at $x = 100$. Then use your polynomial to estimate $\sqrt{105}$.

3. What is the largest possible error that could have occurred in your estimate of $\sqrt{105}$?

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) $\int_{1}^{\infty} \frac{6 + \cos x}{x^{0.99}} \, dx$

(b) $\int_{1}^{\infty} \frac{4x^3 - 2x^2}{2x^4 + x + 1} \, dx$
5. The probability density function (pdf) of the length (in minutes) of phone calls on a wireless network is given by \( f(x) = ke^{-0.2x} \) where \( x \) is the number of minutes. Note that the domain is \( x \geq 0 \) since we can’t have a negative number of minutes.

(a) What must be the value of \( k \)?

(b) What fraction of calls last more than 3 minutes?