Problem 1. (10 points) Evaluate the antiderivative

$$\int \frac{2x^3 + 5x^2 - 3x + 11}{x^2 + x - 2} \, dx.$$

Since the numerator’s degree is not less than the denominator’s, start by long division:

$$\begin{array}{r}
2x + 3 \\
\hline
x^2 + x - 2) 2x^3 + 5x^2 - 3x + 11 \\
\quad - 2x^3 - 2x^2 + 4x \\
\hline \quad 3x^2 + x + 11 \\
\quad - 3x^2 - 3x + 6 \\
\hline \quad - 2x + 17 \\
\end{array}$$

Therefore,

$$\frac{2x^3 + 5x^2 - 3x + 11}{x^2 + x - 2} = 2x + 3 + \frac{-2x + 17}{x^2 + x - 2}.$$

Before integrating, we should decompose the last term into partial fractions. The denominator factors:

$$\frac{-2x + 17}{x^2 + x - 2} = \frac{-2x + 17}{(x + 2)(x - 1)} = \frac{A}{x + 2} + \frac{B}{x - 1}$$

And recombining denominators, we get the identity

$$\frac{-2x + 17}{(x + 2)(x - 1)} = \frac{A(x - 1) + B(x + 2)}{(x + 2)(x - 1)}$$

This leads to the system of equations

$$\begin{cases}
A + B = -2 \\
-A + 2B = 17
\end{cases}$$

Adding the two equations eliminates $A$ and shows $3B = 15$, so $B = 5$. Then using the first equation we obtain $A = -7$.

Finally, we have rewritten our integral:

$$\int \frac{2x^3 + 5x^2 - 3x + 11}{x^2 + x - 2} \, dx = \int 2x + 3 + \frac{-7}{x + 2} + \frac{5}{x - 1} \, dx$$

$$= x^2 + 3x - 7 \ln |x + 2| + 5 \ln |x - 1| + C$$

$$= x^2 + 3x + \ln \left| \frac{(x - 1)^5}{(x + 2)^7} \right| + C$$
Problem 2. (10 points) Use integration by parts to evaluate the antiderivative
\[ \int x^3 e^x \, dx. \]

Remember, \(e^x\) does not have a symbolic antiderivative.

The hint reminds us that whatever we do, we won’t be able to use \(e^x\) as an antiderivative \((v)\) part. We do, however, want to choose our derivative \((u)\) part to be polynomial, so that we end up trading for an easier integral.

One suggestion is to split up the integrand creatively by leaving one \(x\) attached to the exponential:

\[ \int x^3 e^x \, dx = \int x^2 xe^x \, dx \]

Then the substitution \(u = x^2\) shows the antiderivative \(\int xe^x \, dx = \frac{1}{2} e^x\). We can use this fact to integrate by parts:

\[
\begin{align*}
\int xe^x dx &= \frac{1}{2} x^2 e^x - \int e^x dx \\
&= \frac{1}{2} x^2 e^x - \frac{1}{2} e^x + C \\
&= \frac{1}{2} (x^2 - 1) e^x + C.
\end{align*}
\]

Problem 3. (5 points) Evaluate the antiderivative \(\int \sin^3 x \cos^6 x \, dx\) using a trigonometric substitution.

Our goal is to "peel off" either a sine or a cosine to become part of a \(du\) for substitution’s sake. But we ought to steal this power from an odd stack — this way, the remaining powers can be converted using the Pythagorean identity without messy square-roots.

This argues for peeling off one power of \(\sin x\) to set up the substitution \(u = \cos x, \, du = -\sin x \, dx\):

\[
\begin{align*}
\int \sin^3 x \cos^6 x \, dx &= \int \sin^2 x \cos^6 x \sin x \, dx \\
&= -\int (1 - \cos^2 x) u^6 \, du \\
&= -\int (1 - u^2) u^6 \, du \\
&= -\int u^6 - u^8 \, du \\
&= \frac{u^9}{9} - \frac{u^{17}}{17} + C \\
&= \frac{\cos^9 x}{19} - \frac{\cos^{17} x}{17} + C.
\end{align*}
\]