Mathematics 105 — Calculus I

Exam 1

February 13, 2009

Your Name: ___________________________ Solution Guide

There are 6 total problems in this exam. On each problem, you must show all your work, or otherwise thoroughly explain your conclusions. **There is always at least one step preceding a final answer.** Units may be requested for your final answer; a point deduction will apply if they are omitted.

On the portion of the exam marked No Calculator, you will be allowed 30 minutes during which your calculator must be closed and put away. If you finish this section early, you may hand in your work early. However, **only after you hand in the “no calculators“ section will you be permitted to use a calculator.** You may not return to the “no calculator” portion after handing it in.

*Before beginning, ensure your calculator is set to Radians mode.*

You will have 80 minutes to complete this exam.

<table>
<thead>
<tr>
<th>Question</th>
<th>Point Value</th>
<th>Your Score</th>
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<tbody>
<tr>
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Problem 1-NC. (25 points) Use the limit definition of derivative to compute \( f'(x) \) for the function

\[
f(x) = xe^x.
\]

**Hint:** Simplify using properties of exponentials. You will also need to know that \( \lim_{h \to 0} \frac{e^h - 1}{h} = 1. \)

The definition of \( f'(x) \), if it exists, is that it takes the value given by the limit

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)e^{x+h} - xe^x}{h} = \lim_{h \to 0} \frac{xe^{x+h} + he^{x+h} - xe^x}{h} = \lim_{h \to 0} \frac{xe^x + he^{x+h} - xe^x}{h} = \lim_{h \to 0} \frac{xe^x(e^h - 1) + he^{x+h}}{h} = \lim_{h \to 0} xe^x \frac{e^h - 1}{h} + e^x e^h.
\]

Now we just need to determine the behavior of the highlighted terms above. We are given in the problem that the first term approaches 1. The second term does as well, since \( e^h \) is a continuous function of \( h \) and this allows us to substitute \( h = 0 \). Thus

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} xe^x \frac{e^h - 1}{h} + e^x e^h = xe^x + e^x.
\]
Problem 2-NC. (25 points) In this question, you will determine enough properties of the function

\[ f(x) = ex + e^{-x} - e \]

to be able to graph it.

Note: remember, \( e \) is just a number. Its particular value, \( \approx 2.71828 \ldots \) is not important to this problem.

(a) (5 points) What is the vertical intercept of this function?

The vertical intercept is just the value of \( f(0) \):

\[ f(0) = e(0) + e^0 - e = 1 - e. \]

(b) (7 points) Determine on what interval(s) this function is increasing, and on what interval(s) it is decreasing.

To do this, we need to know where the derivative \( f' \) is positive and where it’s negative. We can find this by first determining where \( f' \) is zero:

\[ f'(x) = e - e^{-x} = 0 \]

If \( x \) is very large and positive, \( e^{-x} \approx 0 \) and \( f'(x) \approx e - 0 \) is positive.

If \( x \) is very large and negative, \( e^{-x} \to +\infty \) and \( f'(x) \to e^{-\infty} \) is negative.

\[ x = -1 \]

Decreasing: \((−\infty, -1)\) Increasing: \((-1, \infty)\)

(c) (5 points) Is this function always concave up? Always concave down? Neither? Why?

This depends on the sign of the second derivative. But

\[ f''(x) = e^{-x} \]

is always a positive quantity; thus \( f \) is always concave up.

(d) (8 points) On the axes provided, sketch the graph of \( f \) using your answers to parts (a)—(c).
Problem 1. (25 points) This problem concerns the function

\[ g(t) = \frac{(t - 3)(t^2 - t + 1)}{t^2 - 4t + 3}. \]

(a) (8 points) Determine the domain of this function.

This function will be defined anywhere, so long as its denominator is not zero:

\[ t^2 - 4t + 3 \neq 0 \]
\[ (t - 3)(t - 1) \neq 0 \]
\[ t - 3 \neq 0 \quad \text{and} \quad t - 1 \neq 0 \]
\[ t \neq 3 \quad \text{and} \quad t \neq 1 \]

The domain consists of all real numbers except 1 and 3:

\[ (-\infty, 1) \cup (1, 3) \cup (3, \infty) \quad \text{or} \quad \{ t \in \mathbb{R} : t \neq 1 \text{ and } t \neq 3 \}. \]

(b) (10 points) Using algebra, compute \( \lim_{t \to a} g(t) \) for each value of \( a \) not in the domain of \( g \). Explain what each result means about the continuity of \( g \).

According to the result of part (a), we must compute \( \lim_{t \to 1} g(t) \) and \( \lim_{t \to 3} g(t) \). We begin by factoring the denominator:

\[ g(t) = \frac{(t - 3)(t^2 - t + 1)}{(t - 3)(t - 1)} = \frac{t^2 - t + 1}{t - 1} \quad (t \neq 3) \]

We can use this simpler form to compute the limits we want; after all, limits only see the behavior of \( g(t) \) near the \( t \) values in question.

\[ \lim_{t \to 1} g(t) = \lim_{t \to 1} \frac{t^2 - t + 1}{t - 1} = \frac{1}{0} \quad \rightarrow \pm \infty \]

The graph of \( g(t) \) has a vertical asymptote at \( t = 1 \).

\[ \lim_{t \to 3} g(t) = \lim_{t \to 3} \frac{t^2 - t + 1}{t - 1} = \frac{7}{2} = 3.5 \]

The graph of \( g(t) \) has a hole at \( t = 3 \).

(c) (7 points) At left is a partial graph of \( g(t) \). Fill in the gap, clearly indicating the nature of any discontinuities.
Problem 2. (25 points) A weight is attached to a spring and suspended in a container of motor oil. If it is allowed to oscillate, its vertical position (measured in cm above equilibrium) as a function of time \( t \) in seconds might be given by the function

\[
p(t) = 3e^{-t} \cos t.
\]

(a) (10 points) Complete the data table below, and use your results to estimate the values of \( p'(0.9), p'(1), \) and \( p'(1.1) \). Include units in your answers.

<table>
<thead>
<tr>
<th>( t ) (sec)</th>
<th>0.85</th>
<th>0.9</th>
<th>0.95</th>
<th>1</th>
<th>1.05</th>
<th>1.1</th>
<th>1.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p ) (cm)</td>
<td>0.8462</td>
<td>0.7582</td>
<td>0.6749</td>
<td>0.5963</td>
<td>0.5224</td>
<td>0.4530</td>
<td>0.3880</td>
</tr>
</tbody>
</table>

Estimate the derivatives using average rates of change:

\[
p'(0.9) \approx \frac{p(0.95) - p(0.85)}{0.95 - 0.85} \approx \frac{0.6749 - 0.8462 \text{ cm}}{0.1 \text{ sec}} = -1.713 \text{ cm/sec}
\]

\[
p'(1) \approx \frac{p(1.05) - p(0.95)}{1.05 - 0.95} \approx \frac{0.5963 - 0.6749 \text{ cm}}{0.1 \text{ sec}} = -0.713 \text{ cm/sec}
\]

\[
p'(1.1) \approx \frac{p(1.15) - p(1.05)}{1.15 - 1.05} \approx \frac{0.4530 - 0.5224 \text{ cm}}{0.1 \text{ sec}} = -0.691 \text{ cm/sec}
\]

(b) (10 points) Use your answers to part (a) to estimate \( p''(1) \), with units. What does this answer mean in practical terms?

Again, we’ll use an average rate of change — but not the rate of change of \( p \), rather the rate of change of \( p' \).

\[
p''(1) \approx \frac{p'(1.1) - p'(0.9)}{1.1 - 0.9} \approx \frac{-1.344 - (-1.713) \text{ cm/sec}}{0.2 \text{ sec}} = 1.845 \text{ cm/sec}
\]

(c) (5 points) Is it reasonable, based on your answers, to expect that \( p(t) \) satisfies the differential equation

\[
p'' + 2p' = -2p
\]

Why or why not?

We cannot predict this for all values of \( t \), since this would require us to compute \( p' \) and \( p'' \) — but we don’t know how to do that just yet. (This will require something called the "product rule.")

Instead, let’s see whether our estimates from parts (a) and (b) indicate that this differential equation is satisfied at \( t = 1 \). The left-hand side gives

\[
p''(1) \approx 1.845
\]

\[
+ 2p'(1) \approx 2(-1.525)
\]

\[
p''(1) + 2p'(1) \approx 1.845 + 2(-1.525) \approx -1.205
\]

Meanwhile, the right-hand side is

\[
-2p(1) \approx -2(0.5963) = -1.1926
\]

These numbers are reasonably close to one another; within roundoff and estimation error, the differential equation appears to be satisfied at \( t = 1 \). (Answers may vary.)
Problem 3. (25 points) Shown at left is a graph of the derivative of a function $h$. Use the graph to answer the following questions.

(a) (6 points) On what interval(s) is $h(x)$ decreasing?

$h$ will be decreasing wherever $h'$ is negative:

$$(-2, 0) \cup (0, 3)$$

(b) (6 points) List the $x$ values of all local minimum and local maximum point(s) of $h$, and justify your answers in one sentence.

$h$ will have a local minimum wherever $h'$ switches from negative to positive. This happens at $x = 3$.

$h$ will have a local maximum wherever $h'$ switches from positive to negative. This happens at $x = -2$, even though $h'$ does not exist at $x = -2$!

(c) (6 points) The graph of $h'$ is concave up on the interval $(1.5, 3.0)$. What does this mean about the graph of $h(x)$ on that interval?

The answer: absolutely nothing — at least, nothing immediately visible. The concavity of $h'$ will tell us whether $h''$ is increasing or decreasing, and hence whether the graph of $h$ is getting "more" or "less" concave. This is nearly impossible to spot with the naked eye.

(d) (7 points) Using the axes provided at left, sketch a possible graph of the function $h(x)$.

The vertical scale is not important, only the shape and horizontal location.
Problem 4. (25 points) The latest press booklet for the 2009 Lotus Exige S-240 sports car claims it can accelerate from 0—60 mph in 4.0 seconds flat. A recent test-track run showed that under full throttle, the velocity of the car is modeled by the function

\[ v(t) = 40 \sqrt{t} - 5t, \]

where \( v \) is measured in mph and \( t \) in seconds.

(a) (15 points) According to this model, after how many seconds will the car reach its maximum velocity, and what is the maximum velocity?

Note: do this symbolically, showing your work. You may include a graph or data table if you wish, but your answer must be exact.

We wish to find a local maximum of \( v(t) \) — a point at which \( v'(t) = 0 \) and \( v''(t) \) is negative.

\[
\begin{array}{c|c|c|c}
 t & v'(t) & v''(t) \\
 \hline
 0 & + & 20 \\
 4 & - & \frac{10}{64} < 0 \\
 16 & + & 0
\end{array}
\]

According to this work, the car’s velocity has a local maximum after \( t = 16 \) seconds.

The car’s velocity at this point will be \( v(16) = 40 \sqrt{16} - 5(16) = 80 \) mph.

(b) (10 points) Determine the car’s distance function \( d(t) \) — an antiderivative of its velocity — and use it to find the distance the car traveled during the first 10 seconds of this time trial.

Note: write out the units of the antiderivative in your answer. Convert them if you wish.

An antiderivative of our function \( v(t) \) will be

\[
d(t) = 40 \cdot \frac{t^{3/2}}{3/2} - 5 \cdot \frac{t^2}{2} + C = \frac{80}{3} t^{3/2} - \frac{5}{2} t^2 + C
\]

Does the +C matter? Not if we only care about how far the car travels between \( t = 0 \) and \( t = 10 \), as measured by the difference \( d(10) - d(0) \). (The +C will cancel out.) We may as well drop it, or assume that \( d(0) = 0 \).

We then interpret \( d(t) \) to be the distance the car has traveled since \( t = 0 \), and the total distance during the first 10 seconds of the test will be merely

\[
d(10) = \frac{80}{3} (10)^{3/2} - \frac{5}{2} (10)^2 \approx 593.274 \text{ mph} \cdot \text{sec}
\]

The units of this answer will be the product of the units of input and output of \( v \) — namely, mph·sec.

To convert, we remember that there are 3600 seconds in one hour:

\[
593.274 \frac{\text{mi}}{\text{hr}} \cdot \text{sec} = 593.274 \frac{\text{mi}}{3600 \text{ sec}} \cdot \text{sec} \approx 0.1645 \text{ mi}
\]