1. Suppose the solutions of a matrix equation $Ax = b$ are written in the form $p + v_h$, where $p$ is a particular solution of $Ax = b$ and $v_h$ gives all solutions of the corresponding homogeneous equation $Ax = 0$.

Suppose $b = \begin{bmatrix} 2 \\ -13 \\ 0 \\ 2009 \end{bmatrix}$, $p = \begin{bmatrix} 11 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $v_h = x_2$ and $x_5$ are free.

1a. How many rows does $A$ have? (4)
Since $b$ is a r.c. of the columns of $A$.

1b. Label the columns of $A$ as $c_1, c_2, \ldots, c_p$. Is the set $S = \{c_1, c_2, \ldots, c_p\}$ linearly independent?
Explain in terms of the definition of linear independence.

NOTE WELL! A itself is "right unseen"; we do NOT know the actually entries of $A$.
But we do know there are infinitely many solns to $Ax = 0$ since $x_2$ and $x_5$ are free,
and this means there are ways to write $x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 + x_5c_5 = 0$ besides the
trivial solution $x_1 = x_2 = \ldots = x_5 = 0$.

1c. Write $c_3$ as a linear combination of the other columns, or explain why this cannot be done.
We can do this since it is possible to find a soln of $x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 + x_5c_5 = 0$ in which $x_2 \neq 0$.
For example, take $x_2 = 0$ and $x_5 = 1$ in $v_h$; then
$x_1 = -9x_2 + 7x_5 = 7$; $x_3 = 2x_5 = 2$; $x_4 = 0$ and so
$7c_1 + 0c_2 + 2c_3 + 0c_4 + 1c_5 = 0$; set $c_2$ to get
$\frac{c_1}{7} = \frac{c_2}{2} = \frac{c_3}{2}$ (other answers are possible though different divisors of $x_2$ and $x_5$).

1d. Write $c_4$ as a linear combination of the other columns, or explain why this cannot be done.
This cannot be done. For suppose otherwise. If $c_4 = \alpha_1c_1 + \alpha_2c_2 + \alpha_3c_3 + \alpha_4c_4 + \alpha_5c_5$ for some
scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ [it doesn't matter what they are]
then $c_4 = \frac{\alpha_1}{7}c_1 + \frac{\alpha_2}{2}c_2 + \frac{\alpha_3}{2}c_3 + \alpha_4c_4 + \alpha_5c_5$ represents a
soln of $Ax = 0$ in which the weight of $c_4$ is non-zero. But $v_h$
gives all solns of $Ax = 0$ and in $v_h$, $x_4$ is ALWAYS 0, never -1, a contradiction. So $c_4$ is NOT a L.C. of the other columns.

1e. Show how to express $b$ as a linear combination of all $p$ columns in such a way that none of the weights involved are 0.
This means, find a soln of $Ax = b$ in which none of the weights in $x$ are 0.
Then $x_2 = x_5 = 1$; then $x = p + v_h = \begin{bmatrix} 11 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -9 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$b = 9c_1 + 1c_2 - 9c_3 + 5c_4 + 1c_5$ and none of the weights are 0.
2. Suppose \( T : \mathbb{R}^a \rightarrow \mathbb{R}^z \) is a transformation. Give the definitions of each of the following:

2a. \( T \) is a **linear transformation**.

The following 2 conditions must be met:

\[
\begin{align*}
1 & \quad T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \text{for all } \vec{u}, \vec{v} \in \mathbb{R}^a \\
2 & \quad T(\alpha \vec{u}) = \alpha T(\vec{u}) \quad \text{for all } \vec{u} \in \mathbb{R}^a \text{ & all scalars } \alpha.
\end{align*}
\]

2b. \( T \) is onto \( \mathbb{R}^z \).

\[
T \text{ is onto } \mathbb{R}^z \iff \text{ given ANY } \vec{b} \in \mathbb{R}^z, \text{ there is at least one } \vec{x} \in \mathbb{R}^a \text{ for which } T(\vec{x}) = \vec{b}.
\]

2c. Suppose \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) is defined by \( T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_2x_3 + 2x_1 \\ 0 \\ x_1 + x_2 + x_3 \\ 2x_2 + 7 \end{bmatrix} \). Show by example that \( T \) is not a linear transformation and that it actually fails both parts of the definition in (2a).

\[
\begin{align*}
\text{Does } T\left( \begin{bmatrix} \frac{1}{3} \\ 2 \\ 0 \end{bmatrix} \right) + T\left( \begin{bmatrix} \frac{1}{2} \\ 7 \\ 6 \end{bmatrix} \right) = T\left( \begin{bmatrix} \frac{1}{2} \\ 7 \\ 6 \end{bmatrix} \right) \ ? & \quad \text{No, as follows:} \\
\text{The left side is } T\left( \begin{bmatrix} \frac{1}{3} \\ 2 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} x_2x_3 + 2x_1 \\ 0 \\ x_1 + x_2 + x_3 \\ 2x_2 + 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{3} + \frac{7}{3} \\ 2 \end{bmatrix} \\
\text{and the right side is } T\left( \begin{bmatrix} \frac{1}{2} \\ 7 \\ 6 \end{bmatrix} \right) &= \begin{bmatrix} \frac{1}{2} + 14 \\ 0 \\ 6 + 15 \\ 2 + 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + 14 \\ 0 \\ 6 + 15 \\ 2 + 7 \end{bmatrix} = \begin{bmatrix} 46 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \\
\text{and } \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{3} + \frac{7}{3} \\ 2 \end{bmatrix} \neq \begin{bmatrix} 46 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\end{align*}
\]

\[
\begin{align*}
\text{Does } T\left( 10 \begin{bmatrix} \frac{1}{2} \\ 7 \\ 6 \end{bmatrix} \right) = 10 T\left( \begin{bmatrix} \frac{1}{2} \\ 7 \\ 6 \end{bmatrix} \right) \ ? & \quad \text{No, as follows:} \\
T\left( 10 \begin{bmatrix} \frac{1}{2} \\ 7 \\ 6 \end{bmatrix} \right) = T\left( \begin{bmatrix} 5 \\ 20 \\ 30 \end{bmatrix} \right) = \begin{bmatrix} 620 \\ 60 \\ 60 \\ 97 \end{bmatrix} \quad \text{where as } 10 T\left( \begin{bmatrix} \frac{1}{2} \\ 7 \\ 6 \end{bmatrix} \right) = 10 \begin{bmatrix} \frac{8}{11} \\ \frac{60}{11} \end{bmatrix} = \begin{bmatrix} \frac{80}{11} \\ \frac{600}{11} \end{bmatrix}.
\end{align*}
\]

[Note there are so many examples one can give here.]
3. Suppose \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) is the linear transformation whose standard matrix is \( A = \begin{bmatrix} 5 & 11 & 23 \\ 5 & 14 & 17 \\ 2 & 6 & 6 \\ 3 & 8 & 11 \end{bmatrix} \).

3a. Find \( T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \text{"one copy of \( c_1 \) + one copy of \( c_3 \)."} \)

\[ \begin{bmatrix} 28 \\ 22 \\ 8 \\ 14 \end{bmatrix} \]

3b. Let \( \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \) be a vector in \( \mathbb{R}^4 \). What, if any, conditions on \( b_1, \ldots, b_4 \) must be met to guarantee \( \mathbf{b} \) is in the range of \( T \)? Show any matrices you use in answering this question.

We need to know conditions on \( b_1, \ldots, b_4 \) for which the system represented by \( A\mathbf{x} = \mathbf{b} \) is consistent. A super augmented matrix keeps track of \( b_1, \ldots, b_4 \) as \( A \rightarrow \text{RREF}(A) \):

\[
\begin{bmatrix}
5 & 11 & 23 & b_1 & b_2 & b_3 & b_4 \\
5 & 14 & 17 & 0 & 0 & 0 & 0 \\
2 & 6 & 6 & 0 & 0 & 0 & 0 \\
3 & 8 & 11 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & -1/2 & 1/2 & 0 & 0 \\
0 & 1 & 0 & -1/2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

shows the system is \( \text{consistent} \iff \left\{ \begin{array}{l}
0 = b_1 + \frac{1}{2} b_2 - 4 b_4 \\
0 = b_2 - b_3 - b_4
\end{array} \right. \)

3c. Suppose \( \mathbf{d} = \begin{bmatrix} 3 \\ 12 \\ d_3 \\ d_4 \end{bmatrix} \). Use the conditions in (3b) to find all values of \( d_3 \) and \( d_4 \) for which \( \mathbf{d} \) is in the range of \( T \). (Note you will be setting up a little linear system, and you should use our linear algebra techniques to solve it).

we need

\[
\begin{align*}
0 &= 3 + \frac{1}{2} d_4 - 4 d_4 \\
0 &= 12 - d_3 - d_4
\end{align*}
\]

The corresponding augmented matrix is:

\[
\begin{bmatrix}
\frac{7}{2} & -4 & -12 \\
-1 & -1 & -12 \\
\end{bmatrix}
\]

which has RREF \( \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 6 \end{bmatrix} \); i.e., \( d_1 = \begin{bmatrix} 3 \\ 12 \\ 6 \end{bmatrix} \)

3d. Is \( T \) onto \( \mathbb{R}^4 \)? Explain your answer.

No; unless the conditions in (3b) are met, there will be no \( \mathbf{x} \) s.t. \( T(\mathbf{x}) = \mathbf{b} \).

\( (A\mathbf{x} = \mathbf{b} \text{ will represent an inconsistent system if these conditions are not met}) \)

3e. Is \( T \) one-to-one? Explain your answer.

No. Since \( x_3 \) is free, there are infinitely many solutions to \( T(\mathbf{x}) = \mathbf{0} \).

(Thus it is possible for \( T(\mathbf{u}) = T(\mathbf{v}) \) yet \( \mathbf{u} \neq \mathbf{v} \))
4. Let \( F = \begin{bmatrix} 2 & -3 & w \\ 1 & 5 & -2 \end{bmatrix} \) and \( G = \begin{bmatrix} 8 & x & -6 \\ 7 & -2 & 1 \\ 4 & y & -5 \end{bmatrix} \), and \( P = \begin{bmatrix} 7 & 23 & z \\ q & -12 & 9 \end{bmatrix} \); suppose \( FG = P \).

Show all your work in the following:

4a. Find \( q \). The "row 2, col 1" entry of \( P \) is the "row 2, col 1" entry of \( F \) so we multiply (row 2 of \( F \)) by (col 1 of \( G \)) to find
\[
q = \frac{1\cdot8 + 5\cdot7 - 2\cdot4}{25} = \frac{8 + 35 - 8}{25} = \frac{35}{25} = \frac{7}{5}
\]

4b. Find \( w \). (row 1 of \( F \)) \( \times \) (col 1 of \( G \)) = 7 is the "best" to use, because only the unknown \( w \) appears:
\[
2\cdot8 - 3\cdot7 + 4w = 7 \Rightarrow 16 - 21 + 4w = 7 \Rightarrow 4w = 7 - 16 + 21 = 12 \Rightarrow w = 3
\]

4c. Find \( z \). (row 1 of \( F \)) \( \times \) (col 3 of \( G \)) = \( z \) \( \Rightarrow \) \( z = (2)(-6) + (3)(1) + (w)(-5) \); since \( w = 3 \),
\[
z = -12 - 3 - 15 = -30
\]

4d. Find \( x \) and \( y \). Use linear algebra techniques to solve any system this problem requires.

\[
\begin{align*}
\text{(row 1 of } F \text{)} & \times \text{(col 2 of } G \text{)} = 23 \\
2x + 6 + 3y &= 23 \\
\underline{2x + 3y} &= \underline{17} \\
\text{(row 2 of } F \text{)} & \times \text{(col 2 of } G \text{)} = -12 \\
x - 10 - 2y &= -12 \\
\underline{x - 2y} &= \underline{-2}
\end{align*}
\]

We have 2 equations \& 2 unknowns. So, solve by
\[
\begin{bmatrix} 2 & 3 & 17 \\ 1 & -2 & -2 \end{bmatrix}
\]
\[
\sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}
\]
\[
\Rightarrow x = 4, \ y = 3
\]
5. Suppose that $A$ is a $2 \times 2$ matrix and the following row operations convert $A$ into $I_2$: First, rows 1 and 2 of $A$ are swapped. Then 4 copies of row 1 are subtracted from row 2. Finally, row 2 is multiplied by 5.

5a. What three elementary matrices $E_1$, $E_2$, and $E_3$ represent these three row operations, respectively?

5b. Use the elementary matrices to find $A^{-1}$.

To say these three operations "turn $A$ into $I_2$" means $E_3 \left( E_2 \left( E_1 \left(A\right)\right)\right) = I_2$, so $(E_3E_2E_1)A = I_2$ $\Rightarrow$ $E_3E_2E_1 = A^{-1}$

And $E_3E_2E_1 = \begin{bmatrix} 0 & 1 \\ 5 & -20 \end{bmatrix}$

5c. Find $A$.

$A = (A^{-1})^{-1} = \left(\begin{bmatrix} 0 & 1 \\ 5 & -20 \end{bmatrix}\right)^{-1} = \frac{1}{0(-20)-5.1} \begin{bmatrix} -20 & -1 \\ -5 & 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -20 & -1 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1/5 \\ 1 & 0 \end{bmatrix}$

5d. What is the determinant of $A$?

"ad - bc" \[ \left[ \frac{\text{NOT}}{ad - bc} \frac{1}{1} \right] \] $\Rightarrow$ $4.0 - 1\frac{1}{5} = -\frac{1}{5}$

5e. Is $A$ singular or nonsingular?

Remember, our mnemonic for this was, that to have $\text{det}(A) = 0$ was a "singular" [rare] occurrence.

$\Rightarrow$ $A$ is non-singular.