Math 205 (Winter 2011)
Test 1 (50 points)

Name: Solutions

- Check that you have 7 questions on three pages.
- Show all your work to receive full credit for a problem.

1. (8 points) Hurricanes develop low pressure at their centers that generates high winds. The maximum wind speed \( s \) (in knots) and the central pressure \( p \) of a hurricane are approximately related by the equation \( a + bp = s \). We have the following data on four recent Atlantic hurricanes in the United States.

<table>
<thead>
<tr>
<th></th>
<th>905</th>
<th>920</th>
<th>960</th>
<th>990</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>130</td>
<td>110</td>
<td>80</td>
<td>60</td>
</tr>
</tbody>
</table>

(a) Use the data to write a linear system of four equations which might be used to determine \( a \) and \( b \).

\[
\begin{align*}
a + 905b &= 130 \\
    a + 920b &= 110 \\
    a + 960b &= 80 \\
    a + 990b &= 60
\end{align*}
\]

(b) Is the system you wrote in part (a) consistent? Explain.

Augmented matrix of system in part (a) is

\[
\begin{bmatrix}
1 & 905 & 130 \\
1 & 920 & 110 \\
1 & 960 & 80  \\
1 & 990 & 60
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

Since there is a pivot in the last column, system is inconsistent.
2. (9 points) Suppose $B$ is a $4 \times 4$ matrix with columns $\vec{b}_1, \vec{b}_2, \vec{b}_3,$ and $\vec{b}_4$. The solution of the equation $B\vec{x} = \vec{0}$ is given below in parametric vector form.

$$\vec{x} = x_2 \begin{bmatrix} -1.5 \\ 1 \\ 3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0.5 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$ 

(a) Is $\vec{b}_1$ in Span $\{\vec{b}_2, \vec{b}_3, \vec{b}_4\}$? Explain.

Let $x_2 = 1, x_4 = 0$.

Then $\vec{x} = \begin{bmatrix} -1.5 \\ 1 \\ 3 \\ 0 \end{bmatrix}$ is a solution of $B\vec{x} = \vec{0}$.

So $-1.5\vec{b}_1 + \vec{b}_2 + 3\vec{b}_3 = \vec{0}$.

Thus, $\vec{b}_1$ is a linear combination of $\vec{b}_2, \vec{b}_3, \vec{b}_4$.

So $\vec{b}_1 = \frac{-\vec{b}_2 - 3\vec{b}_3}{-1.5}$.

(b) Suppose $\vec{b}$ is a vector in $\mathbb{R}^4$ such that the equation $B\vec{x} = \vec{b}$ is consistent. How many solutions does the equation have? Explain.

$B\vec{x} = \vec{b}$ has infinitely many solutions.

Solution sets of $B\vec{x} = \vec{0}$ and $B\vec{x} = \vec{b}$ are "parallel".

So $B\vec{x} = \vec{b}$ has infinitely many solutions.

(c) Is $B$ invertible? Explain.

There are free variables in $B\vec{x} = \vec{0}$.

So the RREF of $B$ does not have a pivot in every column. Thus $B$ does not reduce to the identity matrix. Hence $B$ is not invertible.
3. (6 points) Let $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ be the vectors in $\mathbb{R}^2$ shown in the figure.

(a) Give a geometric description of $\text{Span}\{\overrightarrow{b}\}$.

$\text{Span}\{\overrightarrow{b}\}$ is a line in $\mathbb{R}^2$ passing through the origin and $\overrightarrow{b}$.

(b) Is the set $\{\overrightarrow{a}, \overrightarrow{c}\}$ linearly independent? Explain.

$\overrightarrow{c}$ is not a multiple of $\overrightarrow{a}$. So the eqn.

$x_1\overrightarrow{a} + x_2\overrightarrow{c} = \overrightarrow{0}$ has only the trivial soln. Hence the

set $\{\overrightarrow{a}, \overrightarrow{c}\}$ is lin. independent.

(c) Is the set $\{\overrightarrow{a}, \overrightarrow{b}\}$ linearly independent? Explain.

$\overrightarrow{b} = 3\overrightarrow{a}$ (this is an estimate from the figure.)

Thus, $\{\overrightarrow{a}, \overrightarrow{b}\}$ is not lin. ind.

(d) Write a non-trivial solution of the vector equation $x_1\overrightarrow{a} + x_2\overrightarrow{b} + x_3\overrightarrow{c} = \overrightarrow{0}$.

$\overrightarrow{b} = 3\overrightarrow{a}$

so $3\overrightarrow{a} - \overrightarrow{b} = \overrightarrow{0}$ i.e. $3\overrightarrow{a} - \overrightarrow{b} + 0\overrightarrow{c} = \overrightarrow{0}$

$x_1 = 3, x_2 = -1, x_3 = 0$ is a non-trivial solution.
4. (5 points) Suppose $T$ is a transformation given by the formula $T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1^2 + x_2 \\ x_2 \\ x_1 - x_2 \end{bmatrix}$.

(a) What are the domain and codomain of $T$?

**Domain is** $\mathbb{R}^2$.
**Codomain is** $\mathbb{R}^3$.

(b) Show that $T$ is not a linear transformation by providing a counterexample.

One possible counterexample:

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

$T(\mathbf{u}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$. Let $\mathbf{w} = T(\mathbf{u} + \mathbf{v})$.

Thus, $T(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} 9 \\ 3 \\ 3 \end{bmatrix}$.

Thus, $T(\mathbf{u}) + T(\mathbf{v}) \neq T(\mathbf{u} + \mathbf{v})$.

5. (4 points) Suppose the second column of a matrix $B$ is twice the first column and the sum of the first three columns of $B$ is the zero vector. Let $A$ be a matrix such that the product $AB$ is defined.

(a) Show that the second column of $AB$ is twice the first column of $AB$.

Let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ be the first three columns of $B$.

Given: $\mathbf{b}_2 = 2 \mathbf{b}_1$. So $A\mathbf{b}_2 = A(2\mathbf{b}_1) = 2(A\mathbf{b}_1)$.

It second column of $AB = 2$ (first column) of $AB$.

(b) Show that the sum of the first three columns of $AB$ is the zero vector.

First three columns of $AB$ are $A\mathbf{b}_1, A\mathbf{b}_2, A\mathbf{b}_3$.

So sum of first three columns

$= A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3$

$= A(\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3)$

$= A(\mathbf{0})$ (since $\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 = \mathbf{0}$)

$= \mathbf{0}$. 

6. (9 points) Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be a linear transformation such that \( T(\vec{e}_1) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \), \( T(\vec{e}_2) = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \) and \( T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

(a) Find the standard matrix of \( T \).

Let \( A \) be the standard matrix of \( T \). Then columns of \( A \) are \( T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3) \).

So \( A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 1 \end{bmatrix} \)

(b) Find \( T(\vec{v}) \) where \( \vec{v} = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} \).

\[
T(\vec{v}) = A\vec{v} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -3+0+7 \\ 6+0+7 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix} .
\]

(c) Is \( T \) onto? Explain.

\[
A \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} .
\]

Since there is a pivot in every row, \( A\vec{x} = \vec{b} \) is consistent for every \( \vec{b} \) in \( \mathbb{R}^2 \).

So \( T(\vec{x}) = \vec{b} \) is consistent for every \( \vec{b} \) in \( \mathbb{R}^2 \).

Hence \( T \) is onto.
7. (5 points) Let \( \tilde{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \), \( \tilde{a}_2 = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} \), and \( \tilde{a}_3 = \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix} \).

(a) Let \( A = [\tilde{a}_1 \; \tilde{a}_2 \; \tilde{a}_3] \), i.e., \( A \) is the matrix with \( \tilde{a}_1 \), \( \tilde{a}_2 \) and \( \tilde{a}_3 \) as its columns. Find the row echelon form (REF) (NOT RREF) of \( A \). Show all the calculations by hand.

\[
\begin{bmatrix}
1 & -1 & 2 \\
0 & 1 & h \\
0 & 5 & 9
\end{bmatrix}
\xrightarrow{R3 = R3 - 5R2}
\begin{bmatrix}
1 & -1 & 2 \\
0 & 1 & h \\
0 & 0 & 9 - 5h
\end{bmatrix}
\]

(b) Find all possible value(s) of \( h \) such that the set \( \{ \tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \} \) is linearly independent.

The set is lin. ind. if the equation \( A\tilde{x} = \tilde{0} \) has only one solution, i.e., if there are no free variables in \( A\tilde{x} = \tilde{0} \).

From the REF in part (a), we see that there will be a pivot in the third column if
\[
9 - 5h \neq 0
\]

i.e. if \( h \neq \frac{9}{5} \).

Thus, all \( h \) values of \( h \) (except \( \frac{9}{5} \)) make the set lin. ind.