Math 105 Test 1 (75 points)

Name: Solutions

- Check that you have 9 questions on four pages.
- Show all your work to receive full credit for a problem.

1. (9 points) Let \( f(x) = 7 - 4x^2 \). Use the limit definition of the derivative to find \( f'(3) \).

\[
\begin{align*}
f'(3) &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\
&= \lim_{h \to 0} \frac{7 - 4(3+h)^2 - (7-36)}{h} \\
&= \lim_{h \to 0} \frac{7 - 4(9 + 6h + h^2) - 7 + 36}{h} \\
&= \lim_{h \to 0} \frac{-36 - 24h - 4h^2 + 36}{h} \\
&= \lim_{h \to 0} \frac{-24h - 4h^2}{h} \\
&= \lim_{h \to 0} \frac{h(-24 - 4h)}{h} \\
&= \lim_{h \to 0} (-24 - 4h) \\
&= -24 - 0 \\
\end{align*}
\]

So \( f'(3) = -24 \)
2. (9 points) The graph of a function $f(x)$ is given below.

(a) Let $F$ be an antiderivative of $f$. Where in the interval $[-5, 5]$ are the inflection points of $F$? As seen in the graph, $F'$ is decreasing on $[-5, 5]$, so $F$ is concave down on $[-5, 5]$. Hence $F$ has no inflection points in $[-5, 5]$.

(b) Rank the three values $f'(-3), f'(4)$, and 0 in increasing order (from the least to the greatest).

\[ f'(4) < f'(-3) < 0. \]

Tangent line at 4 is steeper than the tangent line at -3 and hence has a smaller negative slope.

(c) Sketch the graph of $f'$ on the axes below. You do not need to show the scale on the vertical axis.

$f$ is decreasing on $[-5, 5]$, so $f' < 0$ on $[-5, 5]$.

$\frac{f'}{f''} > -1$
3. (9 points) Is \( y = \frac{2}{x} - x \) a solution of the differential equation \( xy' + y = \frac{4}{x} \)? Justify your answer.

\[
y = \frac{2}{x} - x = 2x^{-1} - x
\]

\[
y' = -2x^{-2} - 1
\]

Plug in \( y \) and \( y' \) into the differential equation:

\[
x (-2x^{-2} - 1) + (2x^{-1} - x) = \frac{4}{x}
\]

\[
-2x^{-1} - x + 2x^{-1} - x = \frac{4}{x}
\]

\[
-2x \neq \frac{4}{x}
\]

Hence, \( y = \frac{2}{x} - x \) is not a solution of the differential equation.
4. (9 points) Solve the IVP (initial value problem): \( y' - 2\sqrt{x} = \frac{4}{3x^2} + \frac{8}{3} \); \( y(4) = 11 \).

\[
y' = 2\sqrt{x} + \frac{4}{3x^2} + \frac{8}{3}
\]

IE \( y' = 2x^{1/2} + \frac{4}{3}x^{-2} + \frac{8}{3} \)

So \( y = 2 \cdot \frac{x^{3/2}}{3} + \frac{4}{3} \cdot \frac{x^{-1}}{(-1)} + \frac{8}{3}x + C \)

IE \( y = \frac{4}{3}x^{3/2} - \frac{4}{3x} + \frac{8}{3}x + C \).

Plug in \( x = 4 \), \( y = 11 \) to find \( C \).

\[
11 = \frac{4}{3} \cdot 4^{3/2} - \frac{4}{12} + \frac{32}{3} + C
\]

\[
11 = \frac{32}{3} - \frac{1}{3} + \frac{32}{3} + C = \frac{63}{3} + C = 21 + C
\]

\[
C = 11 - 21 = -10
\]

So \( y = \frac{4}{3}x^{3/2} - \frac{4}{3x} + \frac{8}{3}x - 10 \).
5. (10 points) The graph of the derivative of a function $f$ is given below. Use the graph to answer the questions that follow.

![Graph of $f'(x)$]

- Local min of $f' \text{ at } x = -2$
- Local max of $f' \text{ at } x = 2.5$
- Inflection point of $f' \text{ at } x = 0$

(a) What are the stationary points of $f$?

Stationary points are where $f'(x) = 0$.

So $\{x = -2\}$ and $\{x = 5\}$.

(b) Where in the interval $[-6, 6]$ does $f$ have local maximum points?

Check the stationary points:

- $f' + +$ so $x = -2$ is a local max.
- $f' - -$ so $x = 5$ is a local max point.

(c) Where in the interval $[-6, 6]$ is the function $f$ concave up?

$f$ is concave up where $f''$ is increasing.

$f'$ is increasing on $(-2, 2.5)$. So $f$ is concave up on $(-2, 2.5)$.

(d) Let $g(x) = f(x) - 1$. Where in the interval $[-6, 6]$ is the function $g''$ negative?

$g''(x) = f''(x)$. So $g''(x) = f''(x)$. Hence $g'' < 0$ where $f'' < 0$.

$f'' < 0$ where $f'$ is decreasing. So $g'' < 0$ on $(-6, -2)$ and $(2.5, 6)$.

(e) Let $h(x) = -2f(x)$. Find $h'(-4)$.

$h'(x) = -2f'(x)$.

$h'(-4) = -2f'(-4) = -2(-80) = 160$. 

6. (8 points) A balloon is being inflated and the volume $V$ of air in the balloon is given by the formula $V = \frac{4}{3} \pi r^3$ cm$^3$, where $r$ is the radius of the balloon in cm. Include units in your answers.

(a) Find the average rate of change of the volume of air when the radius increases from 2 cm. to 3 cm.

\[
\text{Average rate of change of volume} = \frac{V(3) - V(2)}{3 - 2} = \frac{\frac{4}{3} \pi \cdot 27 - \frac{4}{3} \pi \cdot 8}{1} = \frac{4}{3} \pi (27 - 8) = \frac{76 \pi}{3} \text{ cm}^3/\text{cm}.
\]

(b) Interpret the statement $V'(3) = 36\pi$.

At radius = 3 cm, the volume of air in the balloon is increasing at a rate of $36\pi$ cm$^3$ per cm.
7. (9 points) The graph of a function \( f \) is given below. Use the graph to answer the questions that follow.

(a) Evaluate \( \lim_{{x \to -2^-}} f(x) \) or explain why it does not exist.
\[
\lim_{{x \to -2^-}} f(x) = -3
\]

(b) Evaluate \( \lim_{{x \to -2^+}} f(x) \) or explain why it does not exist.
\[
\lim_{{x \to -2}} f(x) \quad \text{does not exist because} \quad \lim_{{x \to -2^-}} f(x) = -3 \quad \text{and} \quad \lim_{{x \to -2^+}} f(x) = 1
\]

(c) Is \( f \) continuous at \( x = 1 \)? Justify your answer using the definition of continuity.
\[
\lim_{{x \to 1}} f(x) = 3 \quad \text{but} \quad f(1) = 1. \quad \text{So} \quad \lim_{{x \to 1}} f(x) \neq f(1).
\]
Hence \( f \) is not continuous at \( x = 1 \).

(d) Evaluate \( \lim_{{h \to 0}} \frac{f(2+h)-f(2)}{h} \) or explain why it does not exist.
\[
\lim_{{h \to 0}} \frac{f(2+h)-f(2)}{h} = f'(2) = 0 \quad \text{(because the tangent line at 2 is horizontal.)}
\]
8. (6 points) The graph of a function \( g \) along with a line \( L \) is given below. The equation of \( L \) is \( y = -3x + 12 \). Find \( g(2) \) and \( g'(2) \).

At \( x = 2 \), the point on \( L \) is \( y = -3(2) + 12 = 6 \). This point is also on the graph of \( g \). Hence \( g(2) = 6 \).

As seen in the graph, \( L \) is tangent to \( g \) at \( x = 2 \). So \( g'(2) = \text{slope of } L = \boxed{1 - 3} \).

9. (6 points) The table below gives the concentration \( C(t) \) (in milligrams per liter) of a certain chemical in a chemical reaction, \( t \) minutes after the start of the reaction. At what rate is the concentration of the chemical changing at 2 minutes? Include units in your answers.

<table>
<thead>
<tr>
<th>( t ) (min)</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C(t) )</td>
<td>8</td>
<td>5.7</td>
<td>4</td>
<td>2.9</td>
</tr>
</tbody>
</table>

\[
\text{Rate} = C'(2) \approx \frac{C(2) - C(1.5)}{2 - 1.5} = \frac{4 - 5.7}{2 - 1.5}
\]

\[
= \frac{-1.7}{0.5}
\]

\[
= -3.4 \text{ milligrams per liter per minute}
\]

So the concentration is decreasing at a rate of 3.4 milligrams per liter per minute.