MATH 206, Section A
Test # 1
SOLUTIONS

1. (20 points) Given a rectangular point, change it to cylindrical and spherical coordinates.
   \((-\sqrt{3}, -3, -2) \Rightarrow x = -\sqrt{3}, \ y = -3, \ z = -2\)
   a) First of all, \(r = \sqrt{x^2 + y^2} = \sqrt{3 + 9} = \sqrt{12} = 2\sqrt{3}\). Since the point is in the third quadrant (both \(x\) and \(y\) are negative), we know that
     \[
     \theta = \pi + \arctan \frac{y}{x} = \pi + \arctan \frac{-3}{-\sqrt{3}} = \pi + \arctan \frac{\sqrt{3}}{3} = \pi + \frac{\pi}{3} = \frac{4\pi}{3}.
     \]
     Thus cylindrical coordinates: \((2\sqrt{3}, \frac{4\pi}{3}, -2)\).
   b) First of all, \(\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3 + 9 + 4} = \sqrt{16} = 4\). We know \(\theta\) form (a), and
     \[
     \varphi = \arccos \frac{z}{\rho} = \arccos \frac{-2}{4} = \arccos \left(-\frac{1}{2}\right) = \frac{2\pi}{3}.
     \]
     Thus spherical coordinates: \((4, \frac{4\pi}{3}, \frac{2\pi}{3})\).

2. (20 points) Let \(\alpha_1 : x - 2y + z = 1\), \(\alpha_2 : 2x + y + z = 7\), and \(\alpha_3 : x - 5y + 3z = 4\). Let \(\ell\) be the line of intersection of \(\alpha_1\) and \(\alpha_2\). Find parametric equations of \(\ell\) and the angle between \(\ell\) and \(\alpha_3\).
   a) First of all, let us find a point on \(\ell\) : \[
   \begin{cases}
   x - 2y + z = 1 \\
   2x + y + z = 7
   \end{cases}
   \]
   Since we need only a point, we can set, for example, \(z = 0\), then \(x = 1 + 2y\) from the first equation, and plug it into the second one, thus \(2(1 + 2y) + y = 7 \iff 5y = 5\), hence \(y = 1\) and \(x = 3\), therefore, \(P(3, 1, 0) \in \ell\). A direction vector is
     \[
     d = n_1 \times n_2 = \begin{vmatrix}
     1 & j & k \\
     1 & -2 & 1 \\
     2 & 1 & 1
     \end{vmatrix} = -2 \ i + 1 \ j + 2 \ k = -3i + j + 5k.
     \]
   Thus \(\ell : x = 3 - 3t, \ y = 1 + t, \ z = 5t\).
   b) Let \(\theta\) be the angle between \(d\) and \(n_3\), then
     \[
     \cos \theta = \frac{d \cdot n_3}{\|d\| \cdot \|n_3\|} = \frac{-3 + 15}{\sqrt{9 + 25\sqrt{1 + 25 + 9}}^2} = \frac{7}{15} > 0,
     \]
   hence \(\theta \in [0, \frac{\pi}{2}]\), therefore, \(\theta \alpha_3 = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \arccos \left(\frac{d}{n_3}\right)\).

3. (20 points) Find a point on the curve \(r(t) = (2\ln t, 4\arctan t, t^2)\) such that the tangent line at the point is perpendicular to the plane \(\langle x, y, z \rangle = (1, 0, 3) + t(1, 0, -1) + s(0, 1, -1)\). Find the distance between the point and the plane.
   First of all, let us find a normal vector to the plane, it's the cross product of the vectors that span the plane:
     \[
     n = (1, 0, -1) \times (0, 1, -1) = \begin{vmatrix}
     i & j & k \\
     1 & 0 & -1 \\
     0 & 1 & -1
     \end{vmatrix} = 0i - 1j + 1k = \hat{i} + \hat{j} + \hat{k}.
     \]
   Thus we want to find \(t > 0\) corresponding to the point \(P\) such that \(r'(t)\) is proportional to \(n\):
     \[
     r'(t) = \left< \frac{2}{t} - \frac{4}{1 + t^2}, 2t \right> = \lambda \hat{i} + \lambda j + \lambda k \Rightarrow \begin{cases}
     2/t = \lambda \\
     4/(1 + t^2) = \lambda \\
     2t = \lambda
     \end{cases} \Rightarrow \frac{2}{t} = 2t \Rightarrow t = 1 \Rightarrow P(0, \pi, 1).
We know that $Q(1,0,3)$ is on the plane, therefore, $\overrightarrow{QP} = (-1, \pi, -2)$ connects the point and the plane, thus

$\text{dist} = \left| \text{proj}_n \overrightarrow{QP} \right| = \left| \text{comp}_n \overrightarrow{QP} \right| = \frac{n \cdot \overrightarrow{QP}}{|n|} = \frac{-1 + \pi - 2}{\sqrt{1 + 1 + 1}} = \frac{\pi - 3}{\sqrt{3}}$.

4. (20 points) Find the domain of the given function. Sketch the domain. Is the domain open, closed, or neither?

$f(x, y) = \sqrt{y - \ln (-x)} \cdot \ln (-y + \arccos x)$

$$\begin{cases}
-x > 0 & \Rightarrow x < 0 \\
y - \ln (-x) \geq 0 & \Rightarrow y \geq \ln (-x) \\
x \in [-1, 1] & \Rightarrow x \in [-1, 1] \\
y - \arccos x > 0 & \Rightarrow y < \arccos x \\
\end{cases}$$

$\text{dom } f = \{(x,y) \in \mathbb{R}^2 : x \in [-1,0), \ln(-x) \leq y < \arccos x \}$

The domain is neither open nor closed because there are pieces of the boundary which are included and pieces that are excluded. An open set doesn’t include the boundary, while a closed set does include.

5. (20 points) For the given vector-function of three variables

$F(x, y, z) = \left( \ln \left(1 + (xy)^2\right), \arctan \left(xz^2\right) \right)$

find its (total) derivative, the Jacobian matrix at $(-1, 2, 1)$, and the image of the vector $(10, 5, 7)$ under the corresponding linear transformation.

Observe that $F(x, y, z) = (f, g)$ where $f = \ln \left(1 + (xy)^2\right)$ and $g = \arctan \left(xz^2\right)$, the derivative is

$$[DF] = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z}
\end{bmatrix} = \begin{bmatrix}
\frac{2xy^2}{1 + (xy)^2} & \frac{2x^2y}{1 + (xy)^2} & 0 \\
\frac{xz^4}{1 + (xz^2)^2} & 0 & \frac{2xz^3}{1 + (xz^2)^2}
\end{bmatrix} \Rightarrow$$

$JF(-1, 2, 1) = \begin{bmatrix}
\frac{-3}{2} & \frac{3}{2} & 0 \\
\frac{1}{2} & 0 & -1
\end{bmatrix} \to (JF(-1, 2, 1)) \begin{bmatrix}
10 \\
5
\end{bmatrix} = \begin{bmatrix}
\frac{-3}{2} & \frac{3}{2} & 0 \\
\frac{1}{2} & 0 & -1
\end{bmatrix} \begin{bmatrix}
10 \\
5
\end{bmatrix} = \begin{bmatrix}
10 \\
5
\end{bmatrix} = \begin{bmatrix}
-12 \\
-2
\end{bmatrix}$

6. (20 points) Let $F = F(u, v)$ be a function of two variables. Suppose that $u = x + y$ and $v = xy$. Find $\frac{\partial F}{\partial x \partial y}$.

The partial derivative of $F$ with respect to $x$ is

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial F}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial x}$$

Thus the second partial mixed derivative of $F$ once with respect to $x$ and once with respect to $y$ is

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial x} \right) = \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial x} \right) = \frac{\partial^2 F}{\partial u \partial y} \frac{\partial u}{\partial x} + \frac{\partial^2 F}{\partial v \partial y} \frac{\partial v}{\partial x}$$

$$= \frac{\partial F}{\partial u} \frac{\partial^2 F}{\partial u \partial v} x + \frac{\partial F}{\partial v} \frac{\partial^2 F}{\partial v \partial u} y + \frac{\partial F}{\partial u} x + \frac{\partial F}{\partial v} y \frac{\partial^2 F}{\partial v \partial u} x = \frac{\partial^2 F}{\partial u \partial v} x + \frac{\partial^2 F}{\partial v \partial u} y + \frac{\partial^2 F}{\partial u \partial v} x + \frac{\partial^2 F}{\partial v \partial u} y$$

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