Math 105: Review for Exam I - Solutions

1. Let \( f(x) = 3 + \sqrt{x + 5} \).
   (a) What is the natural domain of \( f \)? \([-5, \infty), \) which means all reals greater than or equal to 5
   (b) What is the range of \( f \)? \([3, \infty), \) which means all reals greater than or equal to 3

2. For the graph of \( f \) shown, answer the following.
   (a) Evaluate the following.
      i. \( f'(2) = 0 \)
      ii. \( f(3) = 2 \)
      iii. \( \lim_{x \to 3^-} f(x) = -2 \)
      iv. \( \lim_{x \to 3^+} f(x) = 3 \)
      v. \( \lim_{x \to 3} f(x) \) does not exist
      vi. \( \lim_{x \to -2} f(x) = 0 \)
   (b) Where is \( f \) discontinuous? at \( x = 3 \)
   (c) Where does \( f' \) fail to exist?
      at \( x = -3, -1, 0, 3 \)

3. Let \( f(x) = 3x^2 - 2x \).
   (a) Compute the average rate of change of \( f \) on the interval \([2, 2.1]\).
      \[
      \frac{f(2.1) - f(2)}{2.1 - 2} = \frac{9.03 - 8}{0.1} = 10.3
      \]
   (b) Using the limit definition of the derivative, find \( f'(x) \).
      \[
      f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x + h)^2 - 2(x + h) - (3x^2 - 2x)}{h} = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} = \lim_{h \to 0} \frac{h(6x + 3h - 2)}{h} = \lim_{h \to 0} (6x + 3h - 2) = 6x - 2
      \]
   (c) Find the equation of the tangent line to \( f \) at \( x = 2 \).
      We want \( y = mx + b \). \( m = f'(2) = 6 \cdot 2 - 2 = 10 \), so \( y = 10x + b \).
      When \( x = 2 \), \( y = f(2) = 3 \cdot 2^2 - 2 \cdot 2 = 8 \).
      Thus, \( 8 = 10 \cdot 2 + b \), so \( b = -12 \) and we have \( y = 10x - 12 \).
   (d) How would the derivative of \( g(x) = f(x) + 5 \) compare to \( f'(x) \)?
      The graph of \( y = f(x) + 5 \) is the graph of \( y = f(x) \) shifted vertically by 5 units, but this has no effect on the slope of the graph, so \( g'(x) = f'(x) \).
(e) How would the derivative of $h(x) = 5f(x)$ compare to $f'(x)$?

The graph of $y = 5f(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of 5; this also results in slopes that are 5 times greater at any given $x$-value. Thus, $h'(x) = 5f'(x)$.

Note that we get the same result by considering our derivative rule $\frac{d}{dx}[kf(x)] = kf'(x)$ where $k = 5$.

4. Fill in the table showing the graphical relationships between $f$, $f'$, and $f''$.

<table>
<thead>
<tr>
<th>$f$</th>
<th>positive</th>
<th>negative</th>
<th>increasing</th>
<th>decreasing</th>
<th>concave up</th>
<th>concave down</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'$</td>
<td>$X$</td>
<td>$X$</td>
<td>$positive$</td>
<td>$negative$</td>
<td>$increasing$</td>
<td>$decreasing$</td>
</tr>
<tr>
<td>$f''$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$X$</td>
<td>$positive$</td>
<td>$negative$</td>
</tr>
</tbody>
</table>

5. Given the graph of $f$, sketch a graph of $f'$ and a graph of $F$, an antiderivative of $f$ such that $F(0) = -2$.

Note: The concave up portion in the middle of the graph of $f$ is a perfect parabola, so its derivative ($f'$) is linear; since you don’t know the equation for $f$, your graph of $f'$ may be concave up/down there.
6. Shown below is a graph of $f'$ on its entire domain. The graph is NOT $f$.

At which $x$-value(s)

(a) does $f$ have a stationary point? $c, e, j$
(b) does $f$ have a local max? $e$
(c) does $f$ have a local min? $c$
(d) does $f'$ have a stationary point? $d, h, j$
(e) does $f'$ have a local max? $d, j$
(f) does $f'$ have a local min? $h$

(g) is $f$ greatest? $a$
(h) is $f$ least? $k$
(i) is $f'$ greatest? $d$
(j) is $f'$ least? $a$
(k) is $f''$ greatest? $b$
(l) is $f''$ least? $f$

On what interval(s) is

(a) $f$ increasing? $(c, e)$
(b) $f$ decreasing? $[a, c) \cup (e, k]$ (c) $f'$ increasing? $[a, d) \cup (h, j)$
(d) $f'$ decreasing? $(d, h) \cup (j, k])$
(e) $f$ concave up? $[a, d) \cup (h, j)$
(f) $f$ concave down? $(d, h) \cup (j, k]$)

7. Suppose that $T(t)$ gives the temperature in Lewiston as a function of time. In each of the following situations, determine if the signs of $T, T', T''$ are positive, negative, zero, or unknown.

(a) The temperature is 60 degrees and falling steadily.
   The temperature is 60, so we know $T$ is positive.
   The temperature is falling, so we know $T'$ is negative.
   The temperature is falling steadily, so we know the graph is linear, and $T''$ is zero.

(b) The temperature is rising more and more slowly.
   We don’t know whether the temperature is above or below zero, so the sign of $T$ is unknown.
   The temperature is rising, so we know $T'$ is positive.
   The temperature is rising more and more slowly, so we know the graph of $T$ is concave down, and $T''$ is negative.

(c) The temperature is $-5$ degrees and rising.
   The temperature is $-5$, so we know $T$ is negative.
   The temperature is rising, so we know $T'$ is positive.
   We don’t know the concavity of the graph of $T$, so the sign of $T''$ is unknown.

8. The table below gives some values for a function $f(x)$ whose derivative exists at all $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>5.0</td>
<td>6.2</td>
<td>7.3</td>
<td>8.2</td>
<td>9.0</td>
</tr>
</tbody>
</table>

(a) Estimate $f'(1.05)$.

$$f(1.1) - f(1.0) \over 1.1 - 1.0 = 8.2 - 7.3 \over 1.1 - 1.0 = 9$$
(b) Based on the data, is $f''(1.0)$ positive or negative?

Using the same procedure as in the previous part, we can make the following estimates.

$\quad f'(0.85) \approx 12 \quad f'(0.95) \approx 11 \quad f'(1.05) \approx 9 \quad f'(1.15) \approx 8$

We see from these estimates that $f'(x)$ appears to be decreasing near $x = 1$. If $f'(x)$ is decreasing, then $f''(x)$ is negative (that is, $f(x)$ is concave down).

9. Find the derivatives of the following.

(a) $y = 2 + 3x + 4x^4 + 5x^6$

$y' = 3 + 12x^3 + 30x^5$

(b) $y = \sqrt[6]{x} + \frac{1}{x^6} + \frac{6}{x} + \frac{\pi}{6} + 6^{1/2}$

First, rewrite $y$ to make it easier to apply our derivative rules:

$y = x^{1/6} + x^{-6} + \frac{1}{6} \cdot x + 6x^{-1} + \frac{\pi}{6} + 6^{1/2}$

$y' = \frac{1}{6}x^{-5/6} + (-6)x^{-7} + \frac{1}{6} + (6)(-1)(x^{-2}) + 0 + 0$

If necessary, we can rewrite this using exponent rules: $x^{-a} = 1/x^a$ and $x^{a/b} = \sqrt[b]{a}$.

$$y' = \frac{1}{6\sqrt[6]{x^5}} - \frac{6}{x^7} + \frac{1}{6} - \frac{6}{x^2}$$

10. Find antiderivatives of the following.

(a) $y = \pi + 3x^2$

antiderivative = $\pi x + x^3 + C$

(b) $y = 4x^5 - \frac{1}{x^6} = 4x^5 - x^{-6}$

antiderivative = $\frac{4x^6}{6} - \frac{x^{-5}}{-5} + C = \frac{2x^6}{3} + \frac{1}{5x^5} + C$