1. Let $A = \begin{bmatrix} -6 & -12 & -17 & -7 & 5 \\ 10 & 8 & 9 & 7 & -1 \\ 2 & 0 & 1 & -1 & -1 \\ 3 & 3 & 1 & 5 & 2 \\ -1 & -1 & 3 & 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$, and $d = \begin{bmatrix} m \\ n \end{bmatrix}$. Label the column vectors of $A$ as $c_1, \ldots, c_5$.

FACT: $\begin{bmatrix} -6 & -12 & -17 & -7 & 5 \\ 10 & 8 & 9 & 7 & -1 \\ 2 & 0 & 1 & -1 & -1 \\ 3 & 3 & 1 & 5 & 2 \\ -1 & -1 & 3 & 1 & 2 \end{bmatrix}$ has RREF $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

1A. Use the FACT to find any/all conditions on $b_1, \ldots, b_5$ which are necessary and sufficient for $b$ to be in the span of the column vectors $A$.

The last two rows of the RREF in the fact say the system represented by $\hat{A}\hat{x} = \hat{b}$ is consistent $\iff$

$\begin{cases} 0 = b_1 - 3b_3 + 2b_4 + 6b_5 \\ 0 = b_2 - b_3 - 2b_4 + 2b_5 \end{cases}$

1B. Verify that $c_5$ does indeed satisfy the conditions in 1A.

First condition $0 = 5 - 3(1) + 2(2) - 6(2)$?

$5 + 3 + 4 - 12$ ?

$12 - 12$ ?

$= 0 \checkmark$

Second condition $0 = -1 - (1) - 2(2) + 2(2)$?

$= -1 + 1 - 4 + 4$

$= 0 + 0$

$= 0 \checkmark$

1C. Find all values of $m$ and $n$ for which $d$ is in the span of the column vectors $A$.

Using the conditions from 1A $\begin{cases} 0 = m - 3(1) + 2(2) - 6(1) \\ 0 = n - 3 - 2(2) + 2(1) \end{cases}$ $\iff$ $\begin{cases} 0 = m - 9 + 4 - 6 \\ 0 = n - 3 - 4 + 2 \end{cases}$ $\iff$ $\begin{cases} 0 = m - 11 \\ 0 = n - 5 \end{cases}$ $\iff$ $\begin{cases} m = 11 \\ n = 5 \end{cases}$

1D. Is $c_5$ a linear combination of the first four column vectors of $A$? If so, find an explicit LC; if not explain why not.

One way to decide is to consider the augmented matrix $[c_1 \ c_2 \ c_3 \ c_4 \ c_5]$. No computations necessary, because it’s in the fact $[\text{note the location of the } |]$ does NOT change its RREF!]

so the solutions $x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 + x_5c_5 = c_5$ are $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 - 2x_4 \\ -1 + x_4 \\ \text{free} \\ \text{free} \end{bmatrix}$

if we pick $x_4 = 0$ (say) then $\hat{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

giving $0c_1 + 1c_2 - 1c_3 + 0c_4 = c_5$

and this is one of many ways to express $c_5$ as a LC of $\{c_1, c_2, c_3, c_4\}$

ALTERNATE SOLN:

All solns of $\hat{A}\hat{x} = \hat{0}$ are given by the RREF in the fact, as

$\begin{cases} x_1 = 0 \\ x_2 = -2x_4 - x_5 \\ x_3 = x_4 + x_5 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases}$

Choose $x_5 = 1$ so that $(\text{and } x_4 = 0)$

$0c_1 - 2c_2 + 3c_3 + 0c_4 + 1c_5 = \hat{0}$

is a soln $\hat{x}$ of $x_1\hat{c}_1 + \cdots + x_5\hat{c}_5 = \hat{0}$

in which $c_5$ appears with a non-zero weight.

Note $\hat{x}$ in the 1st solution has 4 entries $x_1 \cdots x_4$, but $\hat{x}$ in the alternate soln has 5! $\{x_1 \cdots x_5\}$

MORE ANSWERS (IE IF)

NEXT PAGE
1E. Is $c_1$ a linear combination of the last four column vectors of $A$? If so, find an explicit LC; if not, explain why not.

**NO.** One way to see this is by finding the REF of \( [\begin{array}{c} \hat{c}_2 & \hat{c}_3 & \hat{c}_4 & \hat{c}_5 & \hat{c}_1 \end{array}] \).

It's \( \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \) and this row shows the system represented by \( \alpha_1 \hat{c}_1 + \alpha_2 \hat{c}_2 + \alpha_3 \hat{c}_3 + \alpha_4 \hat{c}_4 + \alpha_5 \hat{c}_5 = \hat{c}_1 \) is inconsistent.

Here's a better answer:

From the solution of $AX = 0$ (obtained in 1D) we see that in **ANY** col, $x_i$ is always 0. If on the other hand, $\hat{c}_i$ is a LS of $\hat{c}_2, \hat{c}_3, \hat{c}_4, \hat{c}_5$, then

\[
\begin{align*}
\hat{c}_1 &= \alpha_1 \hat{c}_2 + \alpha_2 \hat{c}_3 + \alpha_3 \hat{c}_4 + \alpha_4 \hat{c}_5 \\
0 &= -\hat{c}_1 + \alpha_1 \hat{c}_2 + \cdots + \alpha_4 \hat{c}_5 \quad \text{and this represents a solution } AX = 0
\end{align*}
\]

In which $x_i = -1$, impossible since $x_i$ is always 0.

1F. Is the set $S = \{c_1, \ldots, c_5\}$ linearly independent? Explain using the definition or any theorem we've discussed.

**No.** One equivalent to L.D. is, a set of vectors is L.D. \( \iff \) at least one of them is a LS of the others and we showed such an example in problem 1D.

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**Note** on the REF in 1E: you can get this from the fact by switching cols 1 & 5. The matrix is no longer row equivalent (switching cols. is not allowable for row equivalence).

But the complementary applied to this new matrix would produce

\[
\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\]

as row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, and it shows an inconsistency in the corresponding system.