1. **Find the following.** [Substitution tip: usually let \( u \) = a function that’s “inside” another function, especially if \( du \) (possibly off by a multiplying constant) is also present in the integrand.]

(a) Let \( u = \sqrt{x} \), so \( du = \frac{dx}{2\sqrt{x}} \) and \( 2du = \frac{dx}{\sqrt{x}} \)

Now we'll change the limits.

If \( x = 1 \), then \( u = \sqrt{1} = 1 \) and if \( x = 4 \), then \( u = \sqrt{4} = 2 \).

\[
\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = \int_1^2 e^u \cdot 2 \, du
\]

\[
= 2e^u \bigg|_1^2
\]

\[
= 2e^2 - 2e \ (\approx 9.342)
\]

(b) Let \( u = \cos(5x) \), so \( du = -5\sin(5x) \, dx \) and \( -\frac{du}{5} = \sin(5x) \, dx \).

Now we'll change the limits.

If \( x = \pi \), then \( u = \cos(5 \cdot \pi) = -1 \) and if \( x = 2 \), then \( u = \cos(5 \cdot 2\pi) = 1 \).

\[
\int_{\pi}^{2\pi} \cos^7(5x) \sin(5x) \, dx = \int_{-1}^{1} u^7 \cdot \frac{-du}{5}
\]

\[
= -\frac{1}{5} \int_{-1}^{1} u^7 \, du
\]

\[
= -\frac{1}{5} \frac{u^8}{8} \bigg|_{-1}^{1}
\]

\[
= -\frac{1}{40} (1^8 - (-1)^8)
\]

\[
= 0
\]

(c) Use \( u = x^3 \), so \( du = 3x^2 \, dx \) and \( \frac{du}{3} = x^2 \, dx \).

\[
\int \frac{7x^2}{1 + x^6} \, dx = 7 \int \frac{\frac{du}{3}}{1 + u^2}
\]

\[
= \frac{7}{3} \arctan u + C
\]

\[
= \frac{7}{3} \arctan(x^3) + C
\]
(d) Use \( u = 10 - x \), so \( du = -dx \) and \( dx = -du \).

\[
\int x\sqrt{10-x} \, dx = \int (10 - u)\sqrt{u}(-du) \quad \text{Since } u = 10 - x, \text{ we know } x = 10 - u.
\]
\[
= \int (u - 10)\sqrt{u} \, du
\]
\[
= \int (u^{3/2} - 10u^{1/2}) \, du
\]
\[
= \frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2} + C
\]
\[
= \frac{2}{5}(10 - x)^{5/2} - \frac{20}{3}(10 - x)^{3/2} + C
\]

2. If \( f(x) \) is decreasing and concave up, put the following quantities in ascending order.

\[ L_{100}, R_{100}, T_{100}, M_{100}, \int_a^b f(x) \, dx \]

\[ R_{100} < M_{100} < \int_a^b f(x) \, dx < T_{100} < L_{100} \]

What can you say with certainty about where \( S_{200} \) would fit into your list above?

It would be somewhere between \( M_{100} \) and \( T_{100} \) but we don’t know how it compares to \( \int_a^b f(x) \, dx \).

3. Find the best possible left, right, midpoint, trapezoidal, and Simpson’s approximations to \( \int_4^{12} f(t) \, dt \) given the data in the table below.

<table>
<thead>
<tr>
<th>( t )</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

\[ L_4 = (15 + 11 + 8 + 4)(2) = 76 \]
\[ R_4 = (11 + 8 + 4 + 3)(2) = 52 \]
\[ T_4 = \frac{L_4 + R_4}{2} = 64 \]

We cannot compute \( M_4 \) because it requires the values of \( f \) at \( x = 5, 7, 9, \) and \( 11 \). Instead, we do \( M_2 \).

\[ M_2 = (11 + 4)(4) = 60 \]

Now, to find \( S_4 \), we need \( T_2 = \frac{L_2 + R_2}{2} = \frac{(15 + 8)(4) + (8 + 3)(4)}{2} = 68. \)

\[ S_4 = \frac{2M_2 + T_2}{3} = \frac{2(60) + 68}{3} = \frac{188}{3} = 62.67 \]

4. Find bounds for each of the following errors if \( I = \int_2^7 \ln x \, dx \).

(a) \[ |I - L_{100}| \leq \frac{K_1(b-a)^2}{2n} = \frac{\frac{1}{4}(7-2)^2}{2(100)} = \frac{1}{16} \]

\[ K_1 = \max \{|f'(x)|\} \text{ on } [2, 7] = \max \{ \frac{1}{x} \} \text{ on } [2, 7] = \frac{1}{2} \text{ (occurs at } x = 2) \]

(b) \[ |I - T_{100}| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{\frac{1}{4}(7-2)^3}{12(100)^2} = \frac{1}{3840} \]

\[ K_2 = \max \{|f''(x)|\} \text{ on } [2, 7] = \max \{ \frac{1}{x^2} \} \text{ on } [2, 7] = \frac{1}{4} \text{ (occurs at } x = 2) \]
(c) \( |I - M_{100}| \leq \frac{K_2(b - a)^3}{24n^2} = \frac{\frac{1}{2}(7 - 2)^3}{24(100)^2} = \frac{1}{7680} \)
\[ K_2 = \text{same as in previous part} \]

5. If \( I = \int_{2}^{7} \ln x \, dx \), how many subdivisions are required to obtain a trapezoidal sum approximation with error of at most 1/1,000,000?

From part (b) above, we know that
\[ |I - T_n| \leq \frac{K_2(b - a)^3}{12n^2} = \frac{\frac{1}{2}(7 - 2)^3}{12n^2} = \frac{125}{48n^2}. \]
Thus, we want \( \frac{125}{48n^2} \leq \frac{1}{1,000,000} \).
Multiplying each side by 1,000,000n² gives \( \frac{125,000,000}{48} \leq n^2 \).
Taking the square root of each side results in \( \sqrt{\frac{125,000,000}{48}} \leq n \).
Since \( \sqrt{1613.743...} = 1614 \), we must at least 1614 subdivisions.

6. Solve the differential equation \( \frac{dy}{dx} = 2xy + 6x \) if the solution passes through \((0, 5)\).

\[ \frac{dy}{dx} = 2xy + 6x \]
\[ \frac{dy}{dx} = 2x(y + 3) \]
\[ \frac{dy}{y + 3} = 2x \, dx \]
Separate the variables.
\[ \ln |y + 3| = x^2 + C \]
\[ |y + 3| = e^{x^2 + C} \]
\[ y + 3 = \pm C \cdot e^{x^2} \]
\[ y = -3 + Ae^{x^2} \]
Replace \( \pm C \) with \( A \).

Now we use the initial condition \( y(0) = 5 \) to find the value of \( A \).
We have \( 5 = -3 + Ae^{0} \Rightarrow A = 8 \), so the solution is \( y = -3 + 8e^{x^2} \).

7. Write integrals equal to
   (a) the arc length of \( y = x^2 \) on the interval \([1, 5] \)
   \[ \text{arc length of } y = f(x) \text{ on } [a, b] = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx = \int_{1}^{5} \sqrt{1 + (2x)^2} \, dx \approx 24.395 \]
   (b) the area bounded by \( y = x^2 - 8x + 24 \) and \( y = 3x \)
First, find where the curves intersect.

\[
\begin{align*}
  x^2 - 8x + 24 &= 3x \\
  x^2 - 11x + 24 &= 0 \\
  (x - 3)(x - 8) &= 0
\end{align*}
\]

\[\Rightarrow x = 3, x = 8\]

Between \( x = 3 \) and \( x = 8 \), \( y = 3x \) is above \( y = x^2 - 8x + 24 \). (Plug in \( x = 5 \) or graph to check.)

So, the area between them is

\[\int_3^8 [3x - (x^2 - 8x + 24)] \, dx.\]

[This equals \(125/6\).]

8. Consider the region bounded by \( y = \sqrt{x}, y = 0 \), and \( x = 9 \). Write an integral equal to the volume generated if this region is revolved about

(a) the \( x \)-axis

\[
\text{volume of slice} \approx \pi r^2 \Delta x = \pi y^2 \Delta x = \pi (\sqrt{x})^2 \Delta x = \pi x \Delta x
\]

\[\text{total volume} = \pi \int_0^9 x \, dx\]

(b) the line \( x = -1 \)

\[
\text{volume of slice} \approx \pi R^2 \Delta y - \pi r^2 \Delta y = \pi (10^2) \Delta y - \pi (1 + x^2) \Delta y = \pi [100 - (1 + y^2)^2] \Delta y
\]

\[\text{total volume} = \pi \int_{-3}^3 [100 - (1 + y^2)^2] \, dy\]

9. A pyramid has a square base 30 feet to a side and a height of 10 feet. Write integrals equal to

(a) the volume of the pyramid

We slice horizontally, so each slice is a “box” with a square top and bottom and a height (thickness) of \( \Delta h \).

The picture shown below is a vertical cross-section through the center of the pyramid.
Similar triangles: \( \frac{10}{30} = \frac{10 - h}{s} \Rightarrow s = 3(10 - h) \).

Volume of slice \( \approx s^2 \Delta h \approx [3(10 - h)]^2 \Delta h \)

Total volume = \( \int_0^{10} [3(10 - h)]^2 \, dh \)

(b) The work done in pumping all the fluid to a point 5 feet above the pyramid if the pyramid is filled to a height of 8 feet with water (which weighs 62.4 pounds per cubic foot) [Students in the 1:10 section should omit this part.]

We use the same sketch as in the previous part.

\[
\begin{align*}
\text{Volume of slice} & \approx s^2 \Delta h \approx [3(10 - h)]^2 \Delta h & \text{From above.} \\
\text{Weight of slice} & \approx 62.4[3(10 - h)]^2 \Delta h & \text{Weight} = (\text{density})(\text{volume}). \\
\text{Work to lift slice} & \approx 62.4[3(10 - h)]^2 \Delta h(15 - h) & \text{Work} = (\text{force})(\text{distance}); \text{here, force} = \text{weight.} \\
\text{Total work} & = 62.4 \int_0^8 [3(10 - h)]^2 (15 - h) \, dh
\end{align*}
\]