Math 106b
Exam 1
2/4/11

Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification will not receive full credit.

1. (10 pts each) Evaluate the following integrals using integration by substitution.

(a) \[ \int_0^{\pi/6} \frac{\cos x}{1 - \sin x} \, dx \]
\[ \quad \begin{array}{l}
\text{let } u = 1 - \sin x \quad \Rightarrow \quad du = -\cos x \, dx \\
\end{array} \]
\[ \Rightarrow -\int_0^{\pi/6} \frac{\cos x}{1 - \sin x} \, dx = -\int_{1}^{1/2} \frac{1}{u} \, du = -\ln |u| \bigg|_{1}^{1/2} = -\ln 1 - (-\ln 1/2) = \ln 2
\]

(b) \[ \int x\sqrt{2x+1} \, dx \]
\[ \begin{array}{l}
\text{let } u = 2x+1 \quad \Rightarrow \quad x = \frac{1}{2}(u-1) \\
\end{array} \]
\[ \Rightarrow \int x\sqrt{2x+1} \, dx = \frac{1}{4} \int (u-1)\sqrt{u} \, du = \frac{1}{4} \int (u^{3/2} - u^{1/2}) \, du
\]
\[ = \frac{1}{4} \left( \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C
\]
\[ = \frac{1}{10} (2x+1)^{5/2} - \frac{1}{6} (2x+1)^{3/2} + C
\]

2. (10 pts) Find the arc length of a function \( f \) on the interval \([2, 3]\) where its derivative is given by \( f'(x) = \sqrt{x^2 - 1} \).

\[ \text{Arc length} = \int_2^3 \sqrt{1 + (f'(x))^2} \, dx = \int_2^3 \sqrt{1 + \left(\sqrt{x^2 - 1}\right)^2} \, dx
\]
\[ = \int_2^3 \sqrt{1 + x^2 - 1} \, dx = \int_2^3 \sqrt{x^2} \, dx = \int_2^3 x \, dx = \frac{x^2}{2} \bigg|_2^3
\]
\[ = \frac{1}{2} \left( 9 - 4 \right) = \frac{5}{2}
\]
3. Consider the region bounded by the curve \( y = \ln x \), the \( x \)-axis, and the line \( x = 2 \).

(a) (8 pts) Set up, but do not evaluate, the integral which represents the volume of the solid formed by revolving the region around the \( x \)-axis.

\[
\text{Volume of slice} \approx \pi y^2 \Delta x \quad \text{where radius } y = \ln x \quad \Rightarrow \quad V_{\text{slice}} \approx \pi (\ln x)^2 \Delta x
\]

\[
V = \int_1^2 \pi (\ln x)^2 \, dx
\]

(b) (10 pts) Find the volume of the solid formed by revolving the region around the line \( x = -1 \).

\[
V_{\text{slice}} \approx \pi R^2 - \pi r^2 = \pi \left( 9 - (1 + e^y)^2 \right) = \pi \left( 9 - (1 + 2e^y + e^{2y}) \right) - \pi \left( 8 - 2e^y - e^{2y} \right)
\]

\[
V = \pi \int_0^{\ln 2} \left( 8 - 2e^y - e^{2y} \right) \, dy = \pi \left( 8y - 2e^y - \frac{1}{2}e^{2y} \right) \bigg|_0^{\ln 2}
\]

\[
= \pi \left[ \left( 8 \ln 2 - 2e^{\ln 2} - \frac{1}{2}e^{2\ln 2} \right) - \left( 0 - 2e^0 - \frac{1}{2}e^0 \right) \right]
\]

\[
= \pi \left[ \left( 8 \ln 2 - 4 - 2 \right) - \left( -3/2 \right) \right] = \pi \left( 8 \ln 2 - \frac{1}{2} \right)
\]
4. (12 pts) A hemispherical tank with radius 10 feet is filled with water. Set up, but do not evaluate, the integral that finds the amount of work done by a pump in raising all of the water to the level of 6 feet above the top of the tank. Recall that water weighs 62.4 pounds/ft\(^3\).

\[ r \text{ depends upon height of cut} \]

\[ r^2 + y^2 = 10^2 \]
\[ r^2 = 100 - y^2 \]

\[ V_{\text{slice}} \approx \pi r^2 \Delta y = \pi (100 - y^2) \Delta y \text{ ft}^3 \]

\[ \text{Force on slice} \approx (62.4 \text{ lb/ft}^3) \pi (100 - y^2) \Delta y \text{ ft}^3 = 62.4\pi (100 - y^2) \Delta y \text{ lb} \]

\[ \text{Distance to move slice} = 16 - y \]

\[ W = \int_0^{10} 62.4\pi (100 - y^2)(16 - y) \, dy \]
5. Consider the initial value problem \( \frac{dy}{dx} = x(1 + y^2) \) with \( y(0) = 1 \).

(a) (10 pts) Apply Euler's method using step size \( \Delta x = 0.25 \) to estimate \( y(1) \). Round to 4 digits after the decimal. (Note: you are not required to use the table, but regardless of your method, you need to show enough work to justify your answer.)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>1</td>
<td>1.125</td>
<td>1.4082</td>
<td>1.9675</td>
</tr>
<tr>
<td>( \frac{dy}{dx} )</td>
<td>0</td>
<td>0.5</td>
<td>1.1328</td>
<td>2.2373</td>
<td></td>
</tr>
<tr>
<td>( \Delta y )</td>
<td>0</td>
<td>0.125</td>
<td>0.2832</td>
<td>0.5593</td>
<td></td>
</tr>
</tbody>
</table>

\[ y(1) \approx 1.9675 \]

(b) (10 pts) Use separation of variables to find exact the solution to the IVP.

\[
\frac{dy}{dx} = x(1 + y^2)
\]

\[
\int \frac{dy}{1 + y^2} = \int x \, dx
\]

\[ \arctan y = \frac{x^2}{2} + c \]

but \( y(1) = 1 \) so

\[ \arctan 1 = \frac{c}{2} + c \]

\[ \frac{\pi}{4} = c \]

Therefore

\[ y = \tan \left( \frac{x^2}{2} + \frac{\pi}{4} \right) \]
6. Let \( I = \int_0^1 \frac{4}{1 + x^2} \, dx \). If we were to evaluate the integral exactly we would find that \( I = \pi \). Therefore, we can use numerical integration techniques to approximate the value of \( \pi \).

(a) (10 pts) Compute \( L_4 \), the left sum approximation with four subdivisions.

\[
\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = 0.25
\]

\[
L_4 = f(0) \Delta x + f(0.25) \Delta x + f(0.5) \Delta x + f(0.75) \Delta x
\]

\[
= \left(\frac{4}{1+0}\right)(0.25) + \left(\frac{4}{1+(0.25)^2}\right)(0.25) + \left(\frac{4}{1+(0.5)^2}\right)(0.25) + \left(\frac{4}{1+(0.75)^2}\right)(0.25)
\]

\[
\approx 0.125(4 + 3.7647 + 3.2 + 2.56) = 3.3812
\]

So \( \pi \approx 3.3812 \leq L_4 \)

(b) (10 pts) How many subdivisions are required to obtain a left sum approximation with error of at most \( 1/10,000 \)? Recall that the error bound estimates for left sums may be determined using:

\[
|I - L_n| \leq \frac{K_1(b-a)^2}{2n}.
\]

\[
f(x) = 4 \left(1 + x^2\right)^{-1} \Rightarrow f'(x) = -4 \left(1 + x^2\right)^{-2}, 2x = \frac{-8x}{(1 + x^2)^2}
\]

The graph of \( f'(x) \) shows \( |f'(x)| = 2.6 = K_1 \) at the vertex is max of \( |f'(x)| \)

\[
|I - L_n| \leq \frac{2.6 (1-0)^2}{2n} \leq \frac{1}{10000}
\]

\[
26000 \leq 2n
\]

\[n \geq 13000 \]