Math 106: Review for Exam I - SOLUTIONS

1. Find the following. [Substitution tip: usually let \( u \) = a function that’s “inside” another function, especially if \( du \) (possibly off by a multiplying constant) is also present in the integrand.]

(a) Let \( u = \sqrt{x} \), so \( du = \frac{dx}{2\sqrt{x}} \) and \( 2 \, du = \frac{dx}{\sqrt{x}} \)

\[
\int_{1}^{4} e^{\sqrt{x}} \, dx = \int_{1}^{4} e^{u} \cdot 2 \, du \quad \text{If you prefer to switch the limits, use } u = 1 \text{ to } u = 2.
\]

\[
= 2e^{u}\bigg|_{x=1}^{x=4} = 2e^{\sqrt{4}} - 2e^{\sqrt{1}} = 2e^2 - 2e \approx 9.342
\]

(b) Let \( u = \cos(5x) \), so \( du = -5\sin(5x) \) and \( -\frac{du}{5} = \sin(5x) \).

This time, we’ll change the limits:
\( x = \pi \Rightarrow u = \cos(5 \cdot \pi) = -1 \) and \( x = 2\pi \Rightarrow u = \cos(5 \cdot 2\pi) = 1 \)

\[
\int_{\pi}^{2\pi} \cos^{7}(5x) \sin(5x) \, dx = \int_{-1}^{1} u^{7} \cdot -\frac{du}{5} \]

\[
= -\frac{1}{5} \int_{-1}^{1} u^{7} \, du \]

\[
= -\frac{1}{5} \frac{u^{8}}{8} \bigg|_{-1}^{1} \]

\[
= -\frac{1}{40} \left[ \frac{1^8}{8} - \frac{(-1)^8}{8} \right] = 0
\]

(c) Use \( u = x^3 \), so \( du = 3x^2 \, dx \) and \( \frac{du}{3} = x^2 \, dx \).

\[
\int \frac{7x^2}{1 + x^6} \, dx = 7 \int \frac{\frac{du}{3}}{1 + u^2} \]

\[
= \frac{7}{3} \arctan u + C = \frac{7}{3} \arctan(x^3) + C
\]
2. If \( f(x) \) is decreasing and concave up, put the following quantities in ascending order.

\[ L_{100}, R_{100}, T_{100}, M_{100}, \int_a^b f(x) \, dx \]

What can you say with certainty about where \( S_{200} \) would fit into your list above?

It would be somewhere between \( M_{100} \) and \( T_{100} \) but we don’t know how it compares to \( \int_a^b f(x) \, dx \).

3. Suppose \( f(t) \) is the rate of change (in animals per month) of a population \( P(t) \).

(a) What does \( \int_4^{12} f(t) \, dt \) represent in this problem?

It represents the total (or net) change in the number of animals during the time period \([4, 12]\).

(b) Find the best possible left, right, midpoint, trapezoidal, and Simpson’s approximations to \( \int_4^{12} f(t) \, dt \) given the data in the table below.

<table>
<thead>
<tr>
<th>( t )</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

\( L_4 = (15 + 11 + 8 + 4)(2) = 76 \) \hspace{1cm} \( R_4 = (11 + 8 + 4 + 3)(2) = 52 \) \hspace{1cm} \( T_4 = \frac{L_4 + R_4}{2} = 64 \)

We cannot compute \( M_4 \) because it requires the values of \( f \) at \( x = 5, 7, 9, \) and 11. Instead, we do \( M_2 \).

\( M_2 = (11 + 4)(4) = 60 \) Now, to find \( S_4 \), we need \( T_2 = \frac{L_2 + R_2}{2} = \frac{(15 + 8)(4) + (8 + 3)(4)}{2} = 68. \)

\( S_4 = \frac{2M_2 + T_2}{3} = \frac{2(60) + 68}{3} = \frac{188}{3} = 62.5 \)

4. Find bounds for each of the following errors if \( I = \int_2^7 \ln x \, dx \).

(a) \( |I - L_{100}| \leq \frac{K_1(b-a)^2}{2n} = \frac{1}{2} (7 - 2)^2 \), \( K_1 = \max \{ |f'(x)| \} \) on \([2, 7]\) = \( \frac{1}{x} \) on \([2, 7]\) = \( \frac{1}{2} \) (occurs at \( x = 2 \))
(b) \(|I - T_{100}| \leq \frac{K_2(b - a)^3}{12n^2} = \frac{1}{12(100)^2} = \frac{1}{3840}
K_2 = \text{max of } |f''(x)| \text{ on } [2, 7] = \text{max of } \frac{1}{x^2} \text{ on } [2, 7] = \frac{1}{4} (\text{occurs at } x = 2)

(c) \(|I - M_{100}| \leq \frac{K_2(b - a)^3}{24n^2} = \frac{1}{24(100)^2} = \frac{1}{7680}
K_2 = \text{same as in previous part}

5. If \(I = \int_2^7 \ln x \, dx\), how many subdivisions are required to obtain a trapezoidal sum approximation with error of at most \(1/1,000,000\)?

From part (b) above, we know that \(|I - T_n| \leq \frac{K_2(b - a)^3}{12n^2} = \frac{1}{12(100)^2} = \frac{125}{48n^2}\).

Thus, we want \(\frac{125}{48n^2} \leq \frac{1}{1,000,000}\).

Multiplying each side by \(1,000,000n^2\) gives \(\frac{125,000,000}{48} \leq n^2\).

Taking the square root of each side results in \(\sqrt{\frac{125,000,000}{48}} \leq n\).

Since \(\sqrt{\frac{125,000,000}{48}} = 1613.743\ldots\), we must at least 1614 subdivisions.

6. Use Euler’s method with three steps on the differential equation \(\frac{dy}{dt} = y - t\) to estimate \(y(2.5)\) if \(y(1) = 0\).

<table>
<thead>
<tr>
<th>(t)</th>
<th>(y)</th>
<th>(\frac{dy}{dt} \cdot \Delta t = \Delta y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>((-1)(0.5) = -0.5)</td>
</tr>
<tr>
<td>1.5</td>
<td>-0.5</td>
<td>((-2)(0.5) = -1)</td>
</tr>
<tr>
<td>2</td>
<td>-1.5</td>
<td>((-3.5)(0.5) = -1.75)</td>
</tr>
<tr>
<td>2.5</td>
<td>-3.25</td>
<td></td>
</tr>
</tbody>
</table>

So, \(y(2.5) \approx -3.25\) (or -13/4).

7. Solve the differential equation \(\frac{dy}{dx} = 2xy + 6x\) if the solution passes through (0, 5).

\[
\frac{dy}{dx} = 2xy + 6x \\
\frac{dy}{dx} = 2x(y + 3) \\
\frac{dy}{y + 3} = 2x \, dx \\
\int \frac{dy}{y + 3} = \int 2x \, dx \\
\ln|y + 3| = x^2 + C \\
|y + 3| = e^{x^2+C} \\
y + 3 = \pm e^{x^2+C} \\
y = -3 + Ae^{x^2} \\
\]

Exponentiate each side to remove the ln.

\(|y| = z \Rightarrow w = \pm z. \quad |w| = z \Rightarrow w = z. \quad \text{Replace } \pm e^C \text{ with } A.\)

Now we use the initial condition \(y(0) = 5\) to find the value of \(A.\)

We have \(5 = -3 + Ae^0 \Rightarrow A = 8\), so the solution is \(y = -3 + 8e^{x^2}.\)
8. Write integrals equal to

(a) the arc length of \( y = x^2 \) on the interval \([1, 5]\)

\[
\text{arc length of } y = f(x) \text{ on } [a, b] = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_1^5 \sqrt{1 + (2x)^2} \, dx \approx 24.395
\]

(b) the area bounded by \( y = x^2 - 8x + 24 \) and \( y = 3x \)

First, find where the curves intersect.

\[
x^2 - 8x + 24 = 3x \\
x^2 - 11x + 24 = 0 \\
(x - 3)(x - 8) = 0 \\
\Rightarrow x = 3, x = 8
\]

Between \( x = 3 \) and \( x = 8 \), \( y = 3x \) is above \( y = x^2 - 8x + 24 \). (Plug in \( x = 5 \) or graph to check.)

So, the area between them is

\[
\int_3^8 [3x - (x^2 - 8x + 24)] \, dx.
\]

[This equals \( 125/6 \).]

9. Consider the region bounded by \( y = \sqrt{x} \), \( y = 0 \), and \( x = 9 \). Write an integral equal to the volume generated if this region is rotated about

(a) the \( x \)-axis

\[
\text{volume of slice } \approx \pi r^2 \Delta x \\
= \pi y^2 \Delta x \\
= \pi (\sqrt{x})^2 \Delta x \\
= \pi x \Delta x
\]

total volume = \( \pi \int_0^9 x \, dx \)

(b) the line \( x = -1 \)

\[
\text{volume of slice } \approx \pi R^2 \Delta y - \pi r^2 \Delta y \\
= \pi (10^2) \Delta y - \pi (1 + x)^2 \Delta y \\
= \pi [100 - (1 + y^2)^2] \Delta y
\]

total volume = \( \pi \int_0^3 [100 - (1 + y^2)^2] \, dy \)