Math 106: Review for Exam I - SOLUTIONS

1. **Find the following.** [Substitution tip: usually let $u$ = a function that’s “inside” another function, especially if $du$ (possibly off by a multiplying constant) is also present in the integrand.]

(a) Let $u = \sqrt{x}$, so $du = \frac{dx}{2\sqrt{x}}$ and $2du = \frac{dx}{\sqrt{x}}$

Now we’ll change the limits.

If $x = 1$, then $u = \sqrt{1} = 1$ and if $x = 4$, then $u = \sqrt{4} = 2$.

\[
\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = \int_1^2 e^u \cdot 2 \, du \\
= 2e^u|_1^2 \\
= 2e^2 - 2e \ (\approx 9.342)
\]

(b) Let $u = \cos(5x)$, so $du = -5\sin(5x) \, dx$ and $-\frac{du}{5} = \sin(5x) \, dx$.

Now we’ll change the limits.

If $x = \pi$, then $u = \cos(5 \cdot \pi) = -1$ and if $x = 2$, then $u = \cos(5 \cdot 2\pi) = 1$.

\[
\int_\pi^{2\pi} \cos^7(5x) \sin(5x) \, dx = \int_{-1}^1 u^7 \cdot -\frac{du}{5} \\
= -\frac{1}{5} \int_{-1}^1 u^7 \, du \\
= -\frac{1}{5} \frac{u^8}{8}|_{-1}^1 \\
= -\frac{1}{40} (1^8 - (-1)^8) \\
= 0
\]

(c) Use $u = x^3$, so $du = 3x^2 \, dx$ and $\frac{du}{3} = x^2 \, dx$.

\[
\int \frac{7x^2}{1 + x^6} \, dx = 7 \int \frac{du}{1 + u^2} \\
= \frac{7}{3} \arctan u + C \\
= \frac{7}{3} \arctan(x^3) + C
\]
(d) Use \( u = 10 - x \), so \( du = -dx \) and \( dx = -du \).

\[
\int x\sqrt{10 - x} \, dx = \int (10 - u)\sqrt{u} (-du) \quad \text{Since } u = 10 - x, \text{ we know } x = 10 - u.
\]

\[
= \int (u - 10)\sqrt{u} \, du
\]

\[
= \int (u^{3/2} - 10u^{1/2}) \, du
\]

\[
= \frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2} + C
\]

\[
= \frac{2}{5}(10 - x)^{5/2} - \frac{20}{3}(10 - x)^{3/2} + C
\]

2. If \( f(x) \) is decreasing and concave up, put the following quantities in ascending order.

\[ L_{100}, R_{100}, T_{100}, M_{100}, \int_a^b f(x) \, dx \]

\[ R_{100} < M_{100} < \int_a^b f(x) \, dx < T_{100} < L_{100} \]

What can you say with certainty about where \( S_{200} \) would fit into your list above?

It would be somewhere between \( M_{100} \) and \( T_{100} \) but we don’t know how it compares to \( \int_a^b f(x) \, dx \).

3. Find the best possible left, right, midpoint, trapezoidal, and Simpson’s approximations to \( \int_4^{12} f(t) \, dt \) given the data in the table below.

<table>
<thead>
<tr>
<th>( t )</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

\[ L_4 = (15 + 11 + 8 + 4)(2) = 76 \quad R_4 = (11 + 8 + 4 + 3)(2) = 52 \quad T_4 = \frac{L_4 + R_4}{2} = 64 \]

We cannot compute \( M_4 \) because it requires the values of \( f \) at \( x = 5, 7, 9, \) and 11. Instead, we do \( M_2 \).

\[ M_2 = (11 + 4)(4) = 60 \]

Now, to find \( S_4 \), we need \( T_2 = \frac{L_2 + R_2}{2} = \frac{(15 + 8)(4) + (8 + 3)(4)}{2} = 68 \).

\[
S_4 = \frac{2M_2 + T_2}{3} = \frac{2(60) + 68}{3} = \frac{188}{3} = 62.\overline{6}
\]

4. Find bounds for each of the following errors if \( I = \int_2^7 \ln x \, dx \).

(a) \[ |I - L_{100}| \leq \frac{K_1(b-a)^2}{2n} = \frac{\frac{1}{4}(7 - 2)^2}{2(100)} = \frac{1}{16} \]

\[ K_1 = \text{max of } |f'(x)| \text{ on } [2, 7] = \text{max of } \frac{1}{x} \text{ on } [2, 7] = \frac{1}{2} \text{ (occurs at } x = 2) \]

(b) \[ |I - T_{100}| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{\frac{1}{8}(7 - 2)^3}{12(100)^2} = \frac{1}{3840} \]

\[ K_2 = \text{max of } |f''(x)| \text{ on } [2, 7] = \text{max of } \frac{1}{x^2} \text{ on } [2, 7] = \frac{1}{4} \text{ (occurs at } x = 2) \]
(c) \(|I - M_{100}| \leq \frac{K_2(b-a)^3}{24n^2} = \frac{\frac{1}{4}(7-2)^3}{24(100)^2} = \frac{1}{7680}\)

\(K_2 = \text{same as in previous part}\)

5. If \(I = \int_2^7 \ln x \, dx\), how many subdivisions are required to obtain a trapezoidal sum approximation with error of at most \(1/1,000,000\)?

From part (b) above, we know that \(|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{\frac{1}{4}(7-2)^3}{12n^2} = \frac{125}{48n^2}\).

Thus, we want \(\frac{125}{48n^2} \leq \frac{1}{1,000,000}\).

Multiplying each side by 1,000,000 gives \(\frac{125,000,000}{48} \leq n^2\).

Taking the square root of each side results in \(\sqrt{\frac{125,000,000}{48}} \leq n\).

Since \(\sqrt{\frac{125,000,000}{48}} = 1613.743\ldots\), we must at least 1614 subdivisions.

6. Solve the differential equation \(\frac{dy}{dx} = 2xy + 6x\) if the solution passes through \((0, 5)\).

\[
\frac{dy}{dx} = 2xy + 6x \\
\frac{dy}{dx} = 2x(y + 3) \\
\frac{dy}{y+3} = 2x \, dx \\
\int \frac{dy}{y+3} = \int 2x \, dx \\
\ln |y+3| = x^2 + C \\
|y+3| = e^{x^2+C} \\
y + 3 = \pm e^C e^{x^2} \\
y = -3 + Ae^{x^2}
\]

Separate the variables.

Exponentiate each side to remove the ln.

Replace \(\pm e^C\) with \(A\).

Now we use the initial condition \(y(0) = 5\) to find the value of \(A\).

We have \(5 = -3 + Ae^0\) \(\Rightarrow A = 8\), so the solution is \(y = -3 + 8e^{x^2}\).

7. Write integrals equal to

(a) the arc length of \(y = x^2\) on the interval \([1, 5]\)

arc length of \(y = f(x)\) on \([a, b]\) = \(\int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_1^5 \sqrt{1 + (2x)^2} \, dx \approx 24.395\)

(b) the area bounded by \(y = x^2 - 8x + 24\) and \(y = 3x\)
First, find where the curves intersect.

\[ x^2 - 8x + 24 = 3x \]
\[ x^2 - 11x + 24 = 0 \]
\[ (x - 3)(x - 8) = 0 \]
\[ \Rightarrow x = 3, x = 8 \]

Between \( x = 3 \) and \( x = 8 \), \( y = 3x \) is above \( y = x^2 - 8x + 24 \). (Plug in \( x = 5 \) or graph to check.)

So, the area between them is

\[ \int_3^8 [3x - (x^2 - 8x + 24)] \, dx. \]

[This equals \( \frac{125}{6} \).]

8. Consider the region bounded by \( y = \sqrt{x} \), \( y = 0 \), and \( x = 9 \). Write an integral equal to the volume generated if this region is revolved about

(a) the \( x \)-axis

![Diagram](image)

volume of slice \( \approx \pi y^2 \Delta x \)
\[ = \pi y^2 \Delta x \]
\[ = \pi (\sqrt{x})^2 \Delta x \]
\[ = \pi x \Delta x \]

total volume \( = \pi \int_0^9 x \, dx \)

(b) the line \( x = -1 \)

![Diagram](image)

volume of slice \( \approx \pi R^2 \Delta y - \pi r^2 \Delta y \)
\[ = \pi (10^2) \Delta y - \pi (1 + x)^2 \Delta y \]
\[ = \pi [100 - (1 + y^2)^2] \Delta y \]

total volume \( = \pi \int_0^3 [100 - (1 + y^2)^2] \, dy \)
9. A pyramid has a square base 30 feet to a side and a height of 10 feet. Write integrals equal to

(a) the volume of the pyramid

We slice horizontally, so each slice is a “box” with a square top and bottom and a height (thickness) of $\Delta h$.

![Diagram of a pyramid with a slice]

The picture shown below is a vertical cross-section through the center of the pyramid.

![Diagram of a vertical cross-section of a pyramid]

Similar triangles: \( \frac{10}{30} = \frac{10 - h}{s} \Rightarrow s = 3(10 - h) \).

Volume of slice $\approx s^2 \Delta h \approx [3(10 - h)]^2 \Delta h$

Total volume $= \int_0^{10} [3(10 - h)]^2 \, dh$

(b) the work done in pumping all the fluid to a point 5 feet above the pyramid if the pyramid is filled to a height of 8 feet with water (which weighs 62.4 pounds per cubic foot)

We use the same sketch as in the previous part.

Volume of slice $\approx s^2 \Delta h \approx [3(10 - h)]^2 \Delta h$ From above.

Weight of slice $\approx 62.4 [3(10 - h)]^2 \Delta h$ Weight = (density) (volume).

Work to lift slice $\approx 62.4 [3(10 - h)]^2 \Delta h (15 - h)$ Work = (force) (distance); here, force = weight.

Total work $= 62.4 \int_0^8 [3(10 - h)]^2 (15 - h) \, dh$