DO NOT WRITE HERE!

1

2

3

4

5

6

TOTAL

Read the questions CAREFULLY.

Show your work in the space provided.

Make clear what your answers are.

BE NEAT.

Good Luck!
1. Let $R$ be the region shown here, so $R$ is the region to the right of the $y$-axis, below the line $y = 1.5x + 6$ and above the graph of $y = \sqrt{x^3 + 9}$.

1A. Set up and fill in all the formulas for the integral(s) in the form $\int_a^b \frac{\text{"top curve"} - \"bottom curve"}{dx}$ that represent the area of this region. (Do not evaluate the integral(s)).

$$\int_{x=0}^{x=6} (1.5x + 6) - \sqrt{x^3 + 9} \, dx$$

1B. Set up and fill in all the formulas for the integral(s) in the form $\int_a^b \frac{\text{"right curve"} - \"left curve"}{dy}$ that represent the area of this region. (Do not evaluate the integral(s)).

Correctly labeling the sides, expressing $x$ in terms of $y$ yields:

- $x = \frac{y-6}{1.5}$ for $y$ from 6 to 15
- $x = \frac{y}{3}$ for $y$ from 3 to 6

So the left hand curve has formula...

- $x = 0$ for $y$ from 3 to 6
- $x = \frac{y-6}{1.5}$ for $y$ from 6 to 15

And therefore $\int_3^6 \text{ "right" - "left" dy}$ becomes

$$\int_{y=3}^{y=6} \left( \frac{3}{2} \sqrt{y^2 - 9} \right) - (0) \, dy + \int_{y=6}^{y=15} \left( \frac{3}{2} \sqrt{y^2 - 9} \right) - \left( \frac{y-6}{1.5} \right) \, dy$$

Some algebra:

- if $y = \sqrt{x^3 + 9}$
- then $y^2 = x^3 + 9$
- $y^2 - 9 = x^3$
- $x = \frac{3}{2} \sqrt{y^2 - 9}$

- if $y = 1.5x + 6$
- then $y - 6 = 1.5x$
- so $x = \frac{y-6}{1.5}$
2. Let \( R \) be the same region as in problem 1. That is, \( R \) is the region to the right of the \( y \)-axis, below the line \( y = 1.5x + 6 \) and above the graph of \( y = \sqrt{x^3 + 9} \), shown again to the right:

2A. Set up the integral(s) which represent the volume of the solid of revolution obtained by revolving \( R \) around the line \( x = -5 \). (Do not evaluate the integral(s)).

\[
\int_{y=3}^{y=6} \pi \left( \frac{6}{\sqrt{y^3 + 9}} + 5 \right)^2 - 5^2 \, dy + \int_{y=6}^{y=15} \pi \left( \frac{6}{\sqrt{y^3 + 9}} + 5 \right)^2 - \left( \frac{6}{\frac{1}{1.5} + 5} \right)^2 \, dy
\]

2B. Set up the integral(s) which represent the volume of the solid of revolution obtained by revolving \( R \) around the \( x \)-axis. (Do not evaluate the integral(s)).

\[
\pi \int_{x=0}^{x=6} (1.5x + 6)^2 - \left( \sqrt{x^3 + 9} \right)^2 \, dx
\]
3. Consider \( \int_{0.6}^{3} -\ln(x^3 + 1) \, dx \) and the two approximations RHS(20) and TRAP(20) of this integral.

Use appropriate graphs on your calculator to estimate \( K_1 \) and \( K_2 \) to correctly to two digits after the decimal point in the following:

3A: What is value of \( K_1 \) you should use in “theorem 3” to find an error bound for RHS(20)?

\[
K_1 = \frac{b-a}{2n} = \frac{1.59 (3-0.6)^2}{2 \cdot 20} = 0.22896
\]

3B: What is that error bound in (3A)? (to five significant digits)? Show the formula you use in your work.

\[
K_1 = 1.59
\]

3C: What is value of \( K_2 \) you should use in that same theorem to find an error bound for TRAP(20)?

**Useful Info:** \(-\ln(x^3 + 1)'' = \frac{3x(x^3 - 2)}{(x^3 + 1)^2}\)

\[
K_2 = \frac{(b-a)^3}{12n^2}
\]

3D: What is that error bound in (3C)? Again, show the formula as part of your work.

\[
K_2 = 2.18
\]

3E: What’s the smallest value of \( n \) you can choose so that theorem 3 guarantees MID(n) will be within 0.0001 of the exact value of the integral? Show your computations.

\[
\frac{K_2(b-a)^3}{24n^2} \leq 0.0001
\]

we find \( \frac{30.13632}{24 \times 0.0001} \leq n^2 ; 12556.8 \leq n^2 ; 112.05 \leq n \)

so the smallest integer \( n \) is \( n = 113 \)

3F: Find MID(n) for your value of \( n \) to 7 digits after the decimal point.

\[
\text{by calculator program, } M\text{ID}(113) \text{ for } \int_{0.6}^{3} -\ln(x^3 + 1) \, dx
\]

\( \approx -4.4293777 \)
4. A swimming pool has the dimensions shown in the accompanying figure: It’s 12 feet deep at one end, has a rise in the middle, and is 4 feet deep at the shallow end. The curve you see is part of a circle of radius 8’ and its equation is given, assuming that the origin (0,0) is at the corner at the lower left front edge of the pool (marked by 0). The pool is 28’ feet long and 15’ wide. Suppose the pool has 5’ of water in it (at the deep end). Completely set up (but do not evaluate) the integral which represents the amount of work done against gravity by a pump which empties this water from the pool through a pipe that reaches from the bottom of the deep end to 3’ above the top of the pool. Recall that gravity exerts a force of 62.4 pounds on a cubic foot of water.

The water we need to move is in "slices" from y=0 to y=5.

A typical "slice" of water at height y_i has volume dy * 15’ * x_i

where x_i is the x-coordinate here.

Now \((x-18)^2 + y^2 = 8^2\)

\[ \Rightarrow (x-18)^2 = 8^2 - y^2 \]

\[ \Rightarrow x = 18 \pm \sqrt{8^2 - y^2} \]

The + gives x's on THIS side of the circle

The - gives x's on the side we need!

Thus, the FORCE on the slice at height y_i is

\[ dy * 15 * (18 - \sqrt{8^2 - y_i^2}) \text{ 62.4 lbs} \]

The DISTANCE the slice needs to move is \((15 - y_i)\) (see diagram)

so the approx. amount of work done to lift the water at height y_i out of the pool is

\[ dy * 15 * (18 - \sqrt{8^2 - y_i^2}) \text{ 62.4 (15 - y_i) ft-lbs} \]

These same terms appear in a LHS approx of

\[ \int_{y=0}^{y=5} 15 (18 - \sqrt{64 - y_i^2}) \text{ 62.4 (15 - y) dy} \]

which is thus the amount of work done...
5. Find the following integral by substitution. Show all your work!
\[ \int \frac{\cos \sqrt{5x}}{\sqrt{x}} \, dx \]

Let \( u = \sqrt{5x} = (5x)^{\frac{1}{2}} \)
so \( du = \frac{1}{2} (5x)^{-\frac{1}{2}} \cdot 5 \, dx \)
\[ du = \frac{5}{2\sqrt{5x}} \, dx = \frac{5}{2\sqrt{5x}} \, dx \]

So the integral becomes
\[ \frac{2\sqrt{5}}{5} \int \frac{5}{2\sqrt{5x}} \cos \sqrt{5x} \, dx \]
\[ = \frac{2\sqrt{5}}{5} \int \cos u \, du \]
\[ = \frac{2\sqrt{5}}{5} \sin u + C = \frac{2\sqrt{5}}{5} \sin 5x + C \]

6. Find the following integral by substitution. Explicitly show what the integral looks like (in particular, give the new limits) after the substitution is complete.
\[ \int_1^e \frac{1}{x(1 + (\ln x)^2)} \, dx \]

Let \( u = \ln x \)
Then \( du = \frac{1}{x} \, dx \).

If \( x = 1 \), \( u = \ln 1 = 0 \)
If \( x = e \), \( u = \ln e = 1 \)
The integral becomes
\[ \int_{u=0}^{u=1} \frac{1}{1+u^2} \, du \]
\[ = \arctan u \bigg|_0^1 = \arctan 1 - \arctan 0 \]
\[ = \frac{\pi}{4} - \frac{1}{4} \]
\[ = \frac{\pi}{4} \]

\[ \square \]