NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

*Advice*: DON’T spend too much time on a single problem.

<table>
<thead>
<tr>
<th>Problems</th>
<th>Maximum Score</th>
<th>Your Score</th>
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<tr>
<td>1.</td>
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</table>
1. (10 pts.) (a) Evaluate the definite integral
\[ \int_{0}^{\pi/2} e^{\sin x} \cos x \, dx. \]

Let \( u = \sin x \). Then, \( du = \cos x \, dx \). When \( x = 0, u = 0 \) and when \( x = \pi/2, u = 1 \). Thus, we have
\[
\int_{0}^{\pi/2} e^{\sin x} \cos x \, dx = \int_{0}^{1} e^u \, du
\]
\[
= e^u \bigg|_{0}^{1} = e - 1.
\]

(10 pts.) (b) Evaluate the indefinite integral
\[ \int \frac{2x + 2x^3}{\sqrt{1 + x^2}} \, dx. \]

Let \( u = 1 + x^2 \) so that \( du = 2x \, dx \). Now, the numerator \( 2x + 2x^3 \, dx \) becomes \( 2x(1 + x^2) \, dx = u \, du \). Thus,
\[
\int \frac{2x + 2x^3}{\sqrt{1 + x^2}} \, dx = \int \frac{u}{\sqrt{u}} \, du
\]
\[
= \int u^{1/2} \, du
\]
\[
= \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1 + x^2)^{3/2} + C.
\]
2. (20 pts.) Find the area of the region bounded by the curve $y = \ln x$, the $x$-axis, and the line $x = e(2 - y)$. [Hint: sketch a picture of the region by determining the points of intersections among these curves and lines]

First we sketch the graphs for these curves. They intersect at the point $(e, 1)$.

By choosing a horizontal slice, the $x$-coordinate of the left endpoint of the slice is $e^y$ (derived from the curve $y = \ln x$) while the $x$-coordinate of the right endpoint is $e(2 - y)$. Thus, the area of this bounded region is given by

$$\int_0^1 e(2 - y) - e^y \, dy = e(2y - \frac{y^2}{2}) - e^y \bigg|_0^1$$

$$= [e(2 - \frac{1}{2}) - e] - [0 - e^0] = \frac{e}{2} + 1.$$

If you want to use vertical slices, you need to write the area as the sum of two definite integrals as follows.

$$\int_1^e \ln x \, dx + \int_e^{2e} 2 - \frac{x}{e} \, dx.$$
3. (10 pts.) (a) Consider a function $f$ given by the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>-2</td>
<td>1</td>
</tr>
</tbody>
</table>

Find $T_4, M_2$ using the trapezoidal and the mid-point rules respectively for the definite integral $\int_2^4 f(x) \, dx$.

First, we compute the Left-Hand Sum and the Right-Hand Sum. Note that $\Delta x = 0.5$.

$L_4 = (2 + 0 + 4 + (-2))(0.5) = 2 \quad R_4 = (0 + 4 + (-2) + 1)(0.5) = 1.5$

So

$T_4 = \frac{L_4 + R_4}{2} = 1.75$.

Similarly, for the Mid-point Rule, we have only two subintervals each of which has length 1, i.e., $\Delta x = 1$. Now,

$M_2 = (f(2.5) + f(3.5))\Delta x = (0 + (-2))(1) = -2$.

(10 pts.) (b) Recall that the error committed by using the trapezoid approximation $T_n$ is less than or equal to $\frac{K_2(b-a)^3}{12n^2}$ where $|f''(x)| \leq K_2$ for some constant $K_2$ over the interval $[a, b]$. Use this result to give an upper bound for the error committed by $T_8$ for

$I = \int_0^1 e^{x^2} \, dx$.

Here $f(x) = e^{x^2}$. It follows that $f'(x) = e^{x^2}2x$ and $f''(x) = (e^{x^2}2x)(2x) + (e^{x^2})(2)$. Over the interval $[0, 1]$, we have

$|f''(x)| \leq (e^{1.2})(2) + e^{1.2} = 6e$

so we let $K_2 = 6e$. It follows that

$|I - T_8| \leq \frac{6e(1-0)^3}{12 \cdot 8^2} = \frac{e}{128}$. 
4. Let \( R \) be the region bounded by the graph of \( y = (x - 1)^2 \) and the line \( y = x + 1 \).

(15 pts.) Find the volume of the solid obtained from rotating the region around the \( x \)-axis.

The points of intersection \( A \) and \( B \) are \((0, 1)\) and \((3, 4)\) respectively. A typical slice of this solid is a "washer" whose thickness is \( \Delta x \). The radii are measured from the straight line \( y = x + 1 \) and from the curve \( y = (x - 1)^2 \) to the \( x \)-axis respectively. Thus, the volume of the solid is given by

\[
V = \int_0^3 \pi (x + 1)^2 - \pi [(x - 1)^2]^2 \, dx
\]

\[
= \pi \int_0^3 (x^2 + 2x + 1) - (x^4 - 4x^3 + 6x^2 - 4x + 1) \, dx
\]

\[
= \pi \left[ \frac{x^5}{5} + x^4 - \frac{5x^3}{3} + 3x^2 \right]_0^3 = \frac{72\pi}{5}.
\]

(5 pts.) SET UP (do not evaluate) a definite integral representing the arc length of the portion of the curve \( y = (x - 1)^2 \) from \( A \) to \( B \) where \( A \) and \( B \) are the points of intersection.

Here \( f(x) = (x - 1)^2 \) so that \( f'(x) = 2(x - 1) \) and \( [f'(x)]^2 = 4(x^2 - 2x + 1) \). The length of the curve from \( A = (0, 1) \) to \( B = (3, 4) \) is given by

\[
L = \int_0^3 \sqrt{1 + [f'(x)]^2} \, dx = \int_0^3 \sqrt{1 + 4(x^2 - 2x + 1)} \, dx = \int_0^3 \sqrt{4x^2 - 8x + 5} \, dx.
\]
5. (10 pts.) (a) Consider the initial value problem

\[ \frac{dy}{dx} = xy \]

with \( y(0) = -2 \).

Estimate the value \( y(1) \) (when \( x = 1 \)) of the solution using Euler’s method with two steps with initial point \((0, -2)\). DO THIS BY HAND and show all your steps.

There are two subintervals with \( x_0 = 0, x_1 = 0.5 \) and \( x_2 = 1 \). By the Euler’s method, we have

\[ y_1 = y_0 + f(x_0, y_0) \Delta x = (-2) + (x_0y_0)(0.5) = -2 \]

It follows that

\[ y_2 = y_1 + (x_1y_1) \Delta x = (-2) + (0.5)(-2)(0.5) = -2.5. \]

Thus, \( y(1) \approx -2.5 \).

(10 pts.) (b) SET UP (do not evaluate) a definite integral representing the work done in pumping fluid from the cone-shaped tank in the figure to the rim. The fluid has density \( \rho \) (constant) and the depth of the fluid is \( 10 \) m. [gravitational constant is \( g \)]

A typical layer of fluid has weight \( \rho g \pi r^2 \Delta y \). Using similar triangles, \( \frac{r}{y} = \frac{12.5}{20} \) or \( r = \frac{5y}{8} \). The distance to be lifted is \( 20 - y \) and the layer varies from height \( y = 0 \) to \( y = 10 \). Thus, the total work done is given by

\[ W = \int_{0}^{10} \rho g \pi \left( \frac{5y}{8} \right)^2 (20 - y) \, dy. \]