Problem 1. (10 points) Approximating \( \int_0^1 \sin \frac{x}{2} \, dx \) in four different ways gives the following sums:

\[
L_{10} = 0.2208 \quad R_{10} = 0.2688 \quad T_{10} = 0.2448 \quad M_{10} = 0.2449
\]

(a) (5 points) What is the greatest amount of error that \( L_{10} \) can commit for this integral? Answer to four significant figures, and explain your work.

We can use the error bound

\[
|I - L_N| \leq \frac{K(b - a)^2}{2N}
\]

where \( K \) is an upper bound for the derivative \( f' \) on [0, 1].

But \( f'(x) = \frac{1}{2} \cos \frac{x}{2} \), and since cosine is never larger than 1 in absolute value, we have \(-1/2 \leq f'(x) \leq 1/2\). Thus taking \( K = 1/2 \) gives us the bound

\[
|I - L_N| \leq \frac{K(b - a)^2}{2N} = \frac{1/2(1 - 0)^2}{2(10)} = 0.025.
\]

This is the greatest amount of error \( L_{10} \) can commit; it optionally gives us the “trap”

\[
L_{10} - 0.025 = 0.1958 \leq I \leq 0.2458 = L_{10} + 0.025
\]

(b) (5 points) What is the greatest amount of error that \( M_{10} \) can commit for this integral? Answer to four significant figures, and explain your work.

For the midpoint rule we have a new error bound:

\[
|I - M_N| \leq \frac{C(b - a)^3}{24N^2}
\]

where \( C \) is an upper bound for the second derivative \( f'' \) on [0, 1].

But \( f''(x) = -\frac{1}{4} \sin \frac{x}{2} \), and since sine is never larger than 1 in absolute value, we have \(-\frac{1}{4} \leq f''(x) \leq \frac{1}{4}\). Thus taking \( C = \frac{1}{4} \) gives the bound

\[
|I - M_N| \leq \frac{C(b - a)^3}{24N^2} = \frac{\frac{1}{4}(1 - 0)^3}{24(10)^2} \approx 0.0001042.
\]

This is the greatest amount of error \( M_{10} \) can commit; it optionally gives us the ”trap”

\[
M_{10} - 0.0001042 = 0.2448 \leq I \leq 0.2450 = M_{10} + 0.0001042
\]
Problem 2. (10 points) A velocimeter in an experimental car measures the following velocities during a 10-second time trial.

<table>
<thead>
<tr>
<th>$t$ (sec)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$ (m/sec)</td>
<td>0</td>
<td>2.56</td>
<td>9</td>
<td>17.6</td>
<td>27</td>
<td>36</td>
<td>43.6</td>
<td>49</td>
<td>51.8</td>
<td>51.8</td>
<td>49</td>
</tr>
</tbody>
</table>

Using an approximation on 5 subintervals, come up with a conservative estimate (that is, an underestimate) for the total distance the sports car has traveled over these 10 seconds. Explain how you can be sure your answer is an underestimate.

The data do not indicate a function which is always increasing or decreasing, so we cannot use the left- or right-hand rule to get a guaranteed underestimate. However, the data do indicate a function which is concave down; the trapezoid rule will then be an underestimate.

To compute $T_5$, take $N = 5$ and $\Delta x = \frac{10-0}{5} = 2$. Then

$$L_5 = 2 \left( 0 + 9 + 27 + 43.6 + 51.8 \right) = 262.8$$
$$R_5 = 2 \left( 9 + 27 + 43.6 + 51.8 + 49 \right) = 360.8$$
$$T_5 = \frac{L_5 + R_5}{2} = \frac{262.8 + 360.8}{2} = 311.8 \text{ m}$$

Problem 3. (5 points) Using three steps of Euler’s method, calculate an approximate value of $y(7)$ if $y$ is a solution of the initial-value problem

$$y' = \frac{y}{x+y} \quad y(1) = -2.$$

Sketch what you’ve done on the slope field provided.

To go from $x = 1$ to $x = 7$ on three steps, each step will be a width $\Delta x = \frac{7-1}{3} = 2$. Then the following table will help us step through Euler’s method:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$y' = \frac{y}{x+y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>−2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.4</td>
</tr>
<tr>
<td>5</td>
<td>2.8</td>
<td>0.359</td>
</tr>
<tr>
<td>7</td>
<td>3.518</td>
<td></td>
</tr>
</tbody>
</table>